DYNAMIC ANALYSIS OF SDOF SYSTEMS USING MODIFIED ENERGY METHOD

M. Jalili Sadr Abad1, M. Mahmoudi1 and E.H. Dowell2
1Faculty of Civil Engineering, Shahid Rajaee Teacher Training University, P.O. Box 16788-15844, Tehran, Iran
2William Holland Hall, Department of Mechanical Engineering and Materials Science, Duke University, P.O. Box 27708, Durham, U.S.A.

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ABSTRACT

In this research, a new numerical method is presented to evaluate the linear and especially the nonlinear dynamic response of single degree of freedom systems. This technique is based on the simultaneous usage of energy and force equilibrium equations which is called the Modified Energy Method. At first, a simple linear system is selected to illustrate the method in detail with harmonic and earthquake loading. Furthermore, several different nonlinear examples are studied such as: a-Coulomb Friction (nonlinear damping) b-Duffing Oscillator (material nonlinearity) and c-Large-angle Simple Pendulum (geometric nonlinearity). The results of the proposed method are compared with other methods, and it was found that this method is more accurate compared with the 4th order Runge-Kutta method for nonlinear analyses.

Keywords: Dynamic analysis; SDOF systems; material nonlinearity; nonlinear damping; geometric nonlinearity; modified energy method (MEM).

1. INTRODUCTION

Dynamic analysis plays a significant role in civil and mechanical engineering in order to evaluate the time-dependent behavior of structures, and can be categorized as linear or nonlinear. The majority of physical systems are inherently nonlinear and have complex mathematical equations; thus, to overcome this difficulty they are often approximated by a linear model. Sometimes the linear approximations of nonlinear problems are suitable for engineering work [1]. However, under some conditions such as when the structure has large deflections or inelastic displacements, the linear assumption is not sufficient and we have to solve a nonlinear problem. For example, the response of structures deforming into their inelastic range during intense ground shaking is of central importance in earthquake...
Generally, there are two types of nonlinear problems in structural dynamics including: material and/or geometry nonlinearities. Moreover, nonlinear terms due to damping as well can be considered as a type of nonlinearity. In material nonlinearity the relation of force-deformation is not linear, so a nonlinear relation between force and displacement must be defined. Duffing Oscillator which has many applications in various fields of engineering (e.g. see [3], [4]) is used in this study to model the material nonlinearity. Many researchers have worked on the evaluation of the response of this system using different methods. For instance, Nourazar and Mirzabeigy by comparing of Differential Transform Method with 4th order Runge-Kutta method have shown that the DTM can predict the response of system in a limited range of time. And if we take the Laplace Transform before using the Differential Transform method, the solution will be improved; this method is referred as MDTM\(^1\) in literature [5]. Hosen and Chowdhury also recommended a new technique for computing the approximate period of Duffing Oscillators based on using the power series [6]. Furthermore, Younesian et al have studied the frequency analysis of generalized Duffing Oscillators that have the following differential equation:

\[
\ddot{x} + \sum_{i=0}^{n} \alpha_{2i+1}x^{2i+1} = 0
\]  

\(\ddot{x}\) is the acceleration of system and \(\alpha_i\) is an odd coefficient [7]. Some researchers have also investigated the chaotic behavior and stability of these systems [8].

Nonlinear geometry is another type of nonlinear behavior in structural analysis. The motion of a Large-angle simple pendulum is one of the well-known examples in this field. In general, this problem is a 3D nonlinear problem but in many cases, like this study, just the symmetric and in-planar motions are considered to reduce the degrees of freedom and have a SDOF problem. This system is used widely in nonlinear control engineering (e.g. refer to [9]–[14]). Moreover, some researchers have studied the combination of Duffing spring with pendulum problem [15], [16].

Nonlinear damping in the equation of motion is one of the most important types of nonlinear problems. Dry or Coulomb damping is an example of this case, which has a considerable application especially in civil and mechanical engineering [17], [18]. In fact, Dry Friction Dampers are one of the well-known examples in this case which are used to improve the performance of buildings against earthquakes [19]–[21]. In this regard, in some studies the dry friction damping has been compared with viscous damping [22]–[24].

It should be noted that this paper focuses on single degree of freedom systems. Thus the proposed method will be most useful when the dynamics of a complex system are dominated by a single mode or when each mode of a complex system may be treated individually and the coupling between modes can be neglected. Thus systems which respond in resonance are most likely to be well described by the present methods. Of course for complex systems with multiple modes which have significant coupling one in principle use

\(^{1}\) Modified Differential Transform Method
linear Eigen-analysis or nonlinear normal mode theory to find the dominant modes of the system (See Reference [25]).

2. METHODS OF NONLINEAR DYNAMIC ANALYSIS

Often in practical analysis, numerical methods are used to approximate the exact solution. Generally, there are many methods and techniques for numerically solving a nonlinear problem but they can be categorized in three main groups. Direct integration methods including explicit and implicit methods are the first category which are commonly used to determine the dynamic response of the structure; Newmark [26], Wilson [27] and CDM methods are the most famous examples of this category [28]. In this case, Bathe proposed a method based on using the Trapezoidal rule which can better preserve the conservation of energy and momentum during the dynamic analysis vs. the Newmark method [29]. Table 1 shows a summary for some of the numerical step by step integration methods.

<table>
<thead>
<tr>
<th>Methods</th>
<th>Classification</th>
<th>Assumptions</th>
<th>Stability Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>CDM</td>
<td>explicit</td>
<td>( \ddot{x}<em>j = a_i (x</em>{j+1} - 2x_j + x_{j-1}) ), ( \dot{x}<em>j = \frac{a_0}{2} (x</em>{j+1} - x_{j-1}) )</td>
<td>( \Delta t \leq \frac{T_o}{\pi} )</td>
</tr>
<tr>
<td>Newmark</td>
<td>implicit</td>
<td>( \Delta \dot{x} = \frac{\ddot{x}_j + \gamma \Delta \ddot{x}}{a_0} ), ( \Delta x = \frac{\ddot{x}_j}{a_0} + \frac{0.5 \ddot{x}_j + \beta \Delta \ddot{x}}{a_1} )</td>
<td>( \Delta t \leq \frac{T_o}{\pi \sqrt{2(\gamma - 2\beta)}} )</td>
</tr>
<tr>
<td>Wilson</td>
<td>implicit</td>
<td>( \ddot{x}(\tau) = \ddot{x}<em>{j+1} + a_2 (\ddot{x}</em>{j+1} - \ddot{x}_j) ), ( 0 \leq \tau \leq \theta \Delta t )</td>
<td>( \theta \geq 1.37 )</td>
</tr>
</tbody>
</table>

Where, \( a_0 = \frac{1}{\Delta t} \), \( a_1 = \frac{1}{\Delta t^2} \), \( a_2 = \frac{1}{\theta \Delta t} \), \( a_j = \frac{1}{(\theta \Delta t)^j} \), \( j = 0, 1, 2, ..., n \)

Transform methods are the second category, in these methods the real nonlinear problem is mapped to a new space and then it can be solved. Homotopy perturbation transform method is an example in this case [30]. Khan et al [31] in investigation of nonlinear differential equations have been shown that if the Laplace transform be combined with the HPTM the results of this method will be improved. In some studies the modal analysis, which is widely used for linear analysis of structures in earthquake engineering, is modified to apply to nonlinear structures [32]–[35].

In Table 2 some common transformation methods which are used in this category are shown.

---

2 Central Difference Method
3 Homotopy Perturbation Transform Method
Table 2: Definition of transformation methods [5]

<table>
<thead>
<tr>
<th>Transform Methods</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Laplace</td>
<td>[ \int_{0}^{\infty} e^{-st}x(t)dt ]</td>
</tr>
<tr>
<td>Fourier</td>
<td>[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iwt}x(t)dt ]</td>
</tr>
<tr>
<td>Differential</td>
<td>[ \frac{1}{k!} \left[ \frac{d^k x(t)}{dt^k} \right]_{t=0} ]</td>
</tr>
</tbody>
</table>

The last category is using of energy method. Some researchers used the energy balance of system and variational principle (Hamilton rule) in order to evaluate the frequency of a simple dynamic structures [36]–[38]. The applications of these methods are too limited; for example, some types of nonlinear terms in the equation of motion such as nonlinear velocity cannot be considered. In addition, the implementation of these methods on problems often leads to a complex equations; therefore, they cannot be used in practical works. Also, there are other numerical studies in solving the energy equations, for example, Laursen and Chawla [39] worked on a formulation of dynamic contact problems by using an energy conserving algorithm. In this work, the ability of the formulation to produce accurate results where more traditional integration schemes fail was emphasized by the numerical simulations. Moreover, Filiatrault et al. [40] studied the energy balance in nonlinear seismic analysis of structures. Four different simple inelastic two-storey steel plane frames used as structural models, and Newmark-Beta average acceleration method was chosen as the algorithm to integrate the equations of motion. They shown that very small time-step were required for achieving small energy balance error (due to the fact that no iteration procedure was implemented within a time-step), and it was mention that further work is required to determine an acceptable energy balance error to be used for different nonlinear solution algorithms.

It is found that in the presence of large deformation in first category, divergence may occur in long time analysis [41]. Moreover, methods of second category are too complex in many cases are not useful in practical analysis, also they have some limitations and assumptions that may restrict the domain of their usage. In addition, in the third category the methods are too complex and they cannot analyze the variety of nonlinear problems (like nonlinear damping). So in the present paper we introduce a new method which is very simple and also comprehensive so that most nonlinear problems in dynamic structures can be treated.

3. MODIFIED ENERGY METHOD (MEM)

In this research, instead of finding a way to solve the equation of motion, the energy equation together with force equilibrium are used simultaneously. For this purpose, firstly, we derive an integral expression for dynamic energy balance of system. Then, numerical
integration methods are applied on the energy equation to obtain an algebraic expression in terms of velocity. Eventually, the real velocity at each time step will be determined by using the equation of motion.

In physics, energy is defined by the area under the force-displacement curve. Table 2 shows different types of mechanical energies in structural dynamics.

<table>
<thead>
<tr>
<th>Force</th>
<th>Energy</th>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spring ( f_s )</td>
<td>Potential energy</td>
<td>( E_p )</td>
<td>( \int f_s , dx )</td>
</tr>
<tr>
<td>Inertia ( f_i )</td>
<td>Kinetic energy</td>
<td>( E_k )</td>
<td>( \int f_i , dx )</td>
</tr>
<tr>
<td>Damping ( f_d )</td>
<td>Dissipated energy</td>
<td>( E_d )</td>
<td>( \int f_d , dx )</td>
</tr>
<tr>
<td>External force ( F )</td>
<td>Energy of external force</td>
<td>( E_F )</td>
<td>( \int F , dx )</td>
</tr>
</tbody>
</table>

Conservation of energy in a vibrational systems states that:

\[ E_F = E_p + E_K + E_D \]  \hspace{1cm} (2)

The LHS of Eq. (2) represents the energy or work of the external force that can be considered as an input energy. The RHS is the summation of potential, kinetic, and damping energies that arise from work done by system (output energy). According to Table 3, Eq. (2) can be written as:

\[ \int F \, dx = \int f_s \, dx + \int f_i \, dx + \int f_d \, dx \]  \hspace{1cm} (3)

Using the definition of velocity \( dx = v(t) \, dt \), \( E_p \) and \( E_d \) can be expressed in terms of velocity:

\[ \int F(t) \, v(t) \, dt = \int f_s(x) \, dx + \int f_i(x) \, dx + \int f_d(v) \, v(t) \, dt \]  \hspace{1cm} (4)

where, \( x_0 \) is the initial displacement of system.

Applying Newton’s second law, \( \int f_i \, dx \) becomes:

\[ \int_{x_0}^{x} f_i \, dx = \int_{x_0}^{x} \dot{m} \, dx \rightarrow \int m \ddot{x} \, dx \rightarrow \int m \ddot{x} \, dx = \frac{1}{2} \frac{mv^2}{2} - \frac{1}{2} \frac{mv_0^2}{2} \]  \hspace{1cm} (5)
The energy of initial conditions, $E_0$, is defined as follows:

$$E_0 = \int f_s(x) \, dx \bigg|_{x=x_0} + \frac{1}{2} m v_0^2 \quad (6)$$

Accordingly, Eq. (4) can be written in the following form:

$$E_0 + \int_0^1 F(t)v(t) \, dt = \int f_s(x(t)) \, dx + \frac{1}{2} m v_0^2(t) + \int_0^1 f_d(v) v(t) \, dt \quad (7)$$

Eq. (14) expresses the conservation of energy at any time and it can be discretized by using the numerical integration methods. For example, by applying Trapezoidal method Eq. (7), we have:

$$E_0 + \frac{\Delta t}{2} \left[ F(0)v(0) + F(\Delta t)v(\Delta t) \right] = E_p(x) \bigg|_{x=x_0} + \frac{1}{2} m v_0^2(\Delta t) + \frac{\Delta t}{2} \left[ f_s(0)v(0) + f_s(\Delta t)v(\Delta t) \right] \quad (8)$$

Similarly, by using the Simpson method which is more accurate, one can write:

$$E_0 + \frac{\Delta t}{3} \left[ \frac{f_s(0)v(0)}{2} + 4F_s(\Delta_2)v(\Delta_2) + 2F_s(2\Delta_2)v(2\Delta_2) + \ldots + F_s(n\Delta_2)v(n\Delta_2) \right] = E_p(x) + \frac{1}{2} m v_1^2 + \frac{\Delta t}{3} \left[ \frac{f_d(0)v(0)}{2} + 4f_d(\Delta_2)v(\Delta_2) + 2f_d(2\Delta_2)v(2\Delta_2) + \ldots + f_d(n\Delta_2)v(n\Delta_2) \right] \quad (9)$$

It should be mentioned that since the Simpson method needs at least 3 points of integration, Trapezoidal method must be used in the first time step; subsequently, after passing it the Simpson method is used to increase the accuracy of analysis. Furthermore, there is a problem for determination of velocity in the discretized integrals equation because Eq. (8) or (9) are not linear in terms of velocity and a unique velocity cannot be calculated in every time step. For solving this problem and detection of real velocity at each time step, the acceleration is calculated from velocities and by using the equation of motion the real velocity is determined.

### 3.1 Algorithm for modified energy method

1. Definition of dynamic properties of system:
   - Force-deformation relation (stiffness) ($f_s(x)$)
   - Force-velocity relation (damping) ($f_d(v)$)
   - Force-acceleration (mass) ($f = ma$ (Newton’s second law))
Dynamic Analysis of SDOF Systems Using Modified Energy Method

2. Selection of convergence criteria
   - Size of time intervals ($\Delta t$)
   - Maximum allowable error in each time step ($\varepsilon$)

3. Calculation of initial energy (effect of initial conditions)

$$ E_0 = \int_{x_0}^{x_1} f_0(x) dx + \frac{1}{2} m v_0^2 $$  \hspace{1cm} (10)

4. Setting $i=1$ (initial time step)

5. Estimation of displacement (in $t = \Delta t$) using pervious time step, (Euler formula)

$$ x_i \approx v_{i-1} \times \Delta t + x_{i-1} $$  \hspace{1cm} (11)

6. Determination of potential energy

$$ E_p(x_i) = \int_{x_0}^{x_i} f_0(x) dx $$  \hspace{1cm} (12)

7. Rearranging the Energy balance expression

$$ E_K + E_D - E_P = E_0 - E_p $$  \hspace{1cm} (13)

All parameters in the RHS of Eq. (13) are known, and the LHS, which is named $g(v)$, based on Eq. (7) can be written in terms of velocity as follows:

$$ \frac{1}{2} m v^2(t) + \int_0^{i \Delta t} f_0(v) v(t) dt - \int_0^{i \Delta t} F(t) v(t) dt = E_0 - E_p $$  \hspace{1cm} (14)

Discretization of $g(v)$ by using Trapezoidal method leads to an equation in terms of $V_i$

(velocity in $t = i \Delta t$) which is given by $f(v_i) = 0$.

8. Finding the real velocity among all of the roots of $f(v_i)$, i.e.

$$ f(v_i) = 0 \rightarrow v_i^{(1)}, v_i^{(2)}, v_i^{(3)}, ..., v_i^{(N)} $$  \hspace{1cm} (15)

(The subscript of velocity denotes the number of steps and the superscript specifies the number of roots).
- Calculation of acceleration for all roots of $f(v_i)$ using the velocity in previous step

$$a_i = \frac{v_i - v_{i-1}}{\Delta t}$$  \hspace{1cm} (16)

By using equation of motion for each of the roots of $f(v_i)$, we obtain:

$$v_i^{(1)} \rightarrow a_i^{(1)} \rightarrow \Sigma f^{(1)} = ma_i^{(1)} + f_D(v_i^{(1)}) + f_S(x_i) - F(t_i)$$

$$v_i^{(2)} \rightarrow a_i^{(2)} \rightarrow \Sigma f^{(2)} = ma_i^{(2)} + f_D(v_i^{(2)}) + f_S(x_i) - F(t_i)$$

$$\vdots$$

$$v_i^{(N)} \rightarrow a_i^{(N)} \rightarrow \Sigma f^{(N)} = ma_i^{(N)} + f_D(v_i^{(N)}) + f_S(x_i) - F(t_i)$$  \hspace{1cm} (17)

where $\Delta t$ denotes the time of $i$th step.

The real velocity is detected from the equation of motion which better represents the state of equilibrium (i.e. $\Sigma f \approx 0$).

9. Calculate the approximated displacement $(\ddot{x}_i)$ using the velocity calculated in previous step.

$$\ddot{x}_i = v_i \Delta t + x_{i-1}$$  \hspace{1cm} (18)

10. Substitute new $x_i$ (from step 9) into step 6 and iterate among step 6 to step 10.

The convergence criterion for stopping the iteration is given by:

$$\text{If } \left| \frac{\ddot{x}_i - x_i}{x_i} \right| \leq e \Rightarrow \text{Go to step 11}$$  \hspace{1cm} (19)

11. Set $i=i+1$ and Continue the steps 5 to 11 for determining the history of displacement and velocity for each time increment.

- It should be noted that in the first time step the Simpson method cannot be used since it needs at least three points of integration point. Therefore, after passing the first step, Simpson method must be used instead of Trapezoidal method in order to improve the accuracy of calculations (in step 7).

4. NUMERICAL EXAMPLES AND RESULTS

In order to introduce the proposed method in detail, some examples are presented in the
following. Obtained results from proposed method are compared with an analytical solution (if it exists) and the results from other numerical methods. Firstly, a simple linear system is selected to illustrate the method in detail and compares the numerical response with the exact solution; afterwards harmonic and earthquake loading are investigated by MEM for linear systems. Nonlinear examples are then considered including the following: 1-Coulomb Friction (nonlinear damping 2-Duffing Oscillator (material nonlinearity) 3-large-angle pendulum (nonlinear geometry). Table 4 illustrates the definition of parameters which are used in the examples.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Symbols</th>
<th>Parameters</th>
<th>Symbols</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Displacement</td>
<td>$x_0$</td>
<td>Initial Angle</td>
<td>$\theta_0$</td>
</tr>
<tr>
<td>Initial Velocity</td>
<td>$v_0$</td>
<td>Initial Angular Velocity</td>
<td>$\dot{\theta}_0$</td>
</tr>
<tr>
<td>Mass</td>
<td>$m$</td>
<td>Length</td>
<td>$L$</td>
</tr>
<tr>
<td>Coefficient of Linear Stiffness</td>
<td>$k$</td>
<td>Duration of Analysis</td>
<td>$t_d$</td>
</tr>
<tr>
<td>Coefficient of Nonlinear Stiffness</td>
<td>$\alpha$</td>
<td>Number of steps</td>
<td>$n$</td>
</tr>
<tr>
<td>Damping Coefficient</td>
<td>$c$</td>
<td>Max. Allowable Error</td>
<td>$e$</td>
</tr>
<tr>
<td>Damping Ratio</td>
<td>$\xi$</td>
<td>Ground Acceleration</td>
<td>$\ddot{x}_g$</td>
</tr>
</tbody>
</table>

### 4.1 Linear systems

**4.1.1. Example No.1. Free vibration of linear SDOF system with viscous damping.**

![Linear system diagram](image)

$x_0 = 1\text{cm/s}$  
$v_0 = 2\text{cm/s}$  
$m = 1\text{Ns}^2/\text{cm}$  
$k = 4\text{N/cm}$  
$\xi = 0.1$  
$t_d = 10\text{s}$  
$n = 100, 1000$  
$e = 0.001$

Figure 1. Example No.1

Exact solution of this problem is given by:

$$x = \sqrt{x_0^2 + \left(\frac{v_0 + \xi \omega x_0}{\omega_d}\right)^2} e^{-\xi \omega t} \cos \left[ \omega_d t - \tan^{-1} \left(\frac{v_0 + \xi \omega x_0}{\omega_d x_0}\right) \right]$$

(20)
where,
\( x \) = Displacement of system
\( \omega_0 \) = Natural frequency
\( \omega_d \) = Damped natural frequency

Substituting the values of parameter into Eq. (20) yields:

\[
x = 1.49e^{-0.2t} \cos[1.989t - 0.835]
\]

(21)

Now, the proposed method (MEM) is used to find the numerical solution of this example. The initial energy \( (E_i) \) of system is:

\[
E_0 = \frac{1}{2} m v_0^2 + \frac{1}{2} k x_0^2 = 4
\]

(22)

The potential and kinetic energies are also given by:

\[
E_K = \frac{1}{2} m v_{(t)}^2 = \frac{1}{2} v_{(t)}^2
\]

(23)

\[
E_P = \frac{1}{2} k x_{(t)}^2 = 2 x_{(t)}^2
\]

(24)

In addition, damped energy can be written in the following integral:

\[
E_D = 0.4 \int_0^t v_{(t)}^2 dt
\]

(25)

Choosing time steps equal to \( \Delta t = 0.1s \) and using MEM, the results will be as Table 5.

<table>
<thead>
<tr>
<th>t</th>
<th>( x_i )</th>
<th>( E_0 )</th>
<th>( E_p )</th>
<th>( E_k )</th>
<th>( E_d )</th>
<th>( z )</th>
<th>( x_{i+1} )</th>
<th>( x_{exact} )</th>
<th>error %</th>
<th>error &lt;1%</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.2</td>
<td>4</td>
<td>2.88</td>
<td>1</td>
<td>1.16</td>
<td>1.41</td>
<td>1.141</td>
<td>1.17</td>
<td>2.4</td>
<td>×</td>
</tr>
<tr>
<td>0.1</td>
<td>1.141</td>
<td>4</td>
<td>2.6</td>
<td>1.08</td>
<td>0.12</td>
<td>1.47</td>
<td>1.147</td>
<td>1.17</td>
<td>1.9</td>
<td>×</td>
</tr>
<tr>
<td>0.1</td>
<td>1.147</td>
<td>4</td>
<td>2.63</td>
<td>1.31</td>
<td>0.13</td>
<td>1.67</td>
<td>1.162</td>
<td>1.17</td>
<td>0.7</td>
<td>o.k</td>
</tr>
<tr>
<td>0.2</td>
<td>1.324</td>
<td>4</td>
<td>3.5</td>
<td>0.29</td>
<td>0.2</td>
<td>0.77</td>
<td>1.239</td>
<td>1.29</td>
<td>4</td>
<td>×</td>
</tr>
</tbody>
</table>

The previous process is continued by using a computer coding program to the time of \( t=10s \). Fig. 2 depicts the comparison between the numerical results of MEM and exact solution of this problem. Obviously, this figure illustrates that the size of time step (i.e.
\( \Delta t = 0.1 \text{s} \) is not adequate for this example and it is necessary that it becomes smaller.

Figure 2. Comparing the numerical response of MEM with exact solution (\( \Delta t = 0.1 \text{s} \))

To increase the accuracy of analysis, a smaller time step is chosen (\( \Delta t = 0.01 \text{s} \)). As shows in Fig.2, this time step is approximately sufficient for this analysis. In other words, the numerical results approaches to exact solution in this case.

Figure 3. Comparing the numerical response of MEM with exact solution (\( \Delta t = 0.01 \text{s} \))

Once again, it should be noted that the presented example was a simple analysis to illustrate how MEM does work. In the following, we will discuss some advanced examples which consider various types of loading such as harmonic and earthquake loads in order to show the capabilities of the presented method.
4.1.2 Example No.2. forced vibration of linear SDOF system subjected to harmonic loading

\begin{align*}
x_0 &= 1\text{cm/s} \\
v_0 &= 2\text{cm/s} \\
m &= 1\text{Ns}^2/\text{cm} \\
k &= 4\text{N/cm} \\
\xi &= 0.1 \\
p_0 &= 1\text{N} \\
\Omega &= 1\text{rad/s} \\
t_d &= 10\text{s} \\
n &= 2000 \\
e &= 0.001
\end{align*}

Since the stiffness, mass, damping ratio, and initial condition of this system is the same as the previous example; the initial, kinetic, potential, and damping energies will not change and are as mentioned in Example No.1. Besides, the energy of external loading can be expressed as:

\begin{equation}
E_p = \int_0^t F(t)v(t)\,dt = \int_0^t p_0 \cos \Omega t \times v(t)\,dt \quad \text{(26)}
\end{equation}

Fig. 5 gives the comparison between the results of MEM, exact solution, Duhamel and Newmark methods. The constants of $\beta$ and $\gamma$ are selected equal to 0.5 and 0.166 respectively (linear acceleration in each time step is assumed).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example2.png}
\caption{Example No. 2}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{comparison.png}
\caption{Comparing the numerical response of MEM with exact solution, Duhamel and Newmark methods ($\Delta t = 0.005$ s)}
\end{figure}
4.1.3 Example No.3. Vibration of linear one story frame subjected to horizontal earthquake

In this example, because of zero initial conditions, the initial energy is equal to zero and the kinetic, potential, and damping energies of the system are:

\[ E_K = \frac{1}{2} m v^2(t) = \frac{1}{2} v^2(t) \]  \( (27) \)

\[ E_p = \frac{1}{2} k x^2(t) = x^2(t) \]  \( (28) \)

\[ E_D = 0.1414 \int_0^t v^2(t) \, dt \]  \( (29) \)

Moreover, the energy of earthquake loading will be given by:

\[ E_p = \int_0^t F(t) v(t) \, dt = \int_0^t p_{\text{eff}} v(t) \, dt \]

\[ \rightarrow E_p = \int_0^t m \ddot{x}_g(t) \times v(t) \, dt \]  \( (30) \)

in which,

- \( p_{\text{eff}} = \) Effective force of earthquake
- \( \ddot{x}_g = \) Ground acceleration (in this example due to Northridge earthquake)

Fig. 8 shows that there is a small error among the modified energy method with other methods. Although this error is small and it can be neglected in engineering work, it would be concluded that in seismic analyses, while the periods of loading are very small, some errors may be caused in the analysis (i.e. the smaller values of periods \( \Delta t = T / 50 \) have a critical effect for convergence). In general, the value of \( \Delta t = T / 50 \), as a proposed initial time step, has been shown to give good results in these cases, and it can be used in practical seismic analyses.
4.2 Nonlinear systems

4.2.1 Example No. 4. free vibration of nonlinear SDOF system with coulomb damping (nonlinear damping)

\[ x_0 = 0.5 \text{m} \]
\[ v_0 = 0 \]
\[ m = 100 \text{Ns}^2/\text{m} \]
\[ k = 10^6 \text{N/m} \]
\[ \mu = 0.1 \]
\[ t_q = 1 \text{s} \]
\[ n = 1000 \]
\[ e = 0.001 \]
Here the spring is linear, thus the potential energy of system takes a quadratic form as follows:

\[ E_p = \frac{1}{2} k x_{(t)}^2 = 5 \times 10^5 x_{(t)}^2 \]  
(31)

Also, the Kinetic energy of system is:

\[ E_K = \frac{1}{2} m v_{(t)}^2 = 50v_{(t)}^2 \]  
(32)

Since, in this case (Coulomb Damping), damping force depends on the sign of velocity; damping energy may be expressed as:

\[
E_D = \int_{0}^{t} -\mu mg \times \text{sign}(v_{(t)}) \times v_{(t)} \, dt = \begin{cases} 
-\mu mg \int_{0}^{t} v_{(t)} \, dt & \text{for } v_{(t)} > 0 \\
\mu mg \int_{0}^{t} v_{(t)} \, dt & \text{for } v_{(t)} < 0
\end{cases}
\]  
(33)

For convenience, we assume that \(\mu mg = 98.1\), in this example. Now, by using the MEM, the numerical response of system is plotted. Fig. 10 depicts the results of presented method against the exact solution of problem, which is performed by Kelly [1].

![Figure 10. Comparing the numerical response of MEM with direct method (\(\Delta t = 0.001s\)) [1]](image)

It should be noted that the procedure of obtaining the exact solution of Coulomb friction problem is very complex and time consuming because it involves a trial and error procedure.
First, the sign of the velocity is assumed then by solving one of the differential equations of Eq. (33) and imposing the boundary conditions the obtained solution will be valid just for the initial assumption; i.e., it can be used in a limited range of time and would be invalid with changes of the sign of velocity. In other words, in every cycle of oscillation four differential equation together with four boundary condition must be solved. Hence, the direct method is not appropriate in practical analysis and the proposed method can be used with a simple mathematical procedure.

Example No. 5. Free vibration of nonlinear SDOF system with hardening Duffing spring. (Nonlinear material)

By definition of Kinetic and damping energies, we reach the following relation for this example:

\[ \mathbf{E}_K = \frac{1}{2} v(t)^2 \]  
\[ \mathbf{E}_D = 0.2 \int_{0}^{t} v(t)^2 dt \]  

For the problem at hand, the spring force is not linear in terms of displacement. So, the potential energy of system is not quadratic and can be expressed by:

\[ \mathbf{E}_p(x) = \int_{0}^{x} f_s(x) dx = \int_{0}^{x} (kx + \alpha x^3) dx = \frac{1}{2} kx^2 + \frac{\alpha}{4} kx^4 \]

\[ \rightarrow E_p = \frac{1}{2} x^2 + \frac{1}{8} kx^4 \]  

And, the initial energy due to initial condition is:

\[ E_0 = \int_{0}^{x_0} f_s(x) dx + \frac{1}{2} m v_0^2 = \frac{1}{2} x_0^2 + \frac{1}{8} x_0^4 + \frac{1}{2} v_0^2 = 2 \]
Now, the numerical results of this system which are obtained by using the modified energy method, as shown in Fig. 12, is compared with the 4th order Runge-Kutta.

![Comparison of MEM and Runge-Kutta methods](image)

Figure 12. Comparing the numerical response of MEM with Runge-Kutta method ($\Delta t = 0.01s$)

There is no exact mathematical solution to compare the accuracy of these two methods; therefore, here we change the size of time steps, and then the response of two methods is plotted in Fig. 12. Considering a constant time step size, it can be seen that the modified energy method is better converged than the 4th order Runge-Kutta method.

![Accuracy comparison of MEM and Runge-Kutta methods](image)

Figure 13. Comparing the accuracy of MEM and Runge-Kutta methods ($\Delta t = 0.05, 0.01s$)
4.2.2 Example No.6. Free vibration of nonlinear SDOF large-angle simple pendulum. (nonlinear geometry)

\[ \theta_0 = 75^\circ \]
\[ \dot{\theta}_0 = 5 \text{ deg/s} \]
\[ m = 1 \text{ kg} \]
\[ L = 0.248 \text{ m} \]
\[ c = 0.077 \text{ Nms/ rad} \]
\[ t_d = 5 \text{ s} \]
\[ n = 2500 \]
\[ e = 0.001 \]

Figure 14. Example No. 6

Considering the support as datum, the potential energy of system can be written as:

\[
E_p(\theta) = \int_0^\theta M(\theta) d\theta = \int_0^\theta (mgL\sin \theta) d\theta = -mgL\cos \theta \quad \text{mLg = 9.81 \quad L = 0.248} \rightarrow \quad E_p = -2.43288\cos \theta \quad (38)
\]

Also, the Kinetic energy of system is given by:

\[
E_K = \frac{1}{2}mv^2(\theta) = \int_0^{\dot{\theta}_0} E_K = \frac{1}{2}m(L\dot{\theta})^2 \quad \text{mL = 1} \quad L = 0.248 \rightarrow \quad E_K = 0.03752\dot{\theta}^2 \quad (39)
\]

and, damping energy of system in this example can be expressed as:

\[
E_D = \int_0^t c\dot{\theta}^2 dt = \int_0^t c \cdot 0.077 \dot{\theta}^2 dt = 0.077\int_0^t \dot{\theta}^2 dt \quad (40)
\]

In addition, the initial energy due to initial condition would be as follows:

\[
E_0 = -2.43288\cos \theta_0 + 0.03752\dot{\theta}_0^2 \quad (41)
\]

Here, the numerical solution of this example is obtained by using MEM. This response is compared with the 4th order Runge-Kutta method as shown in Fig. 15.

As in the previous example can be performed the accuracy of these two methods can be compared (see Fig. 9).
It can be seen that the proposed method (MEM) using $\Delta t = 0.01$ is closer to the results which consider $\Delta t = 0.002$ in comparison with the results of 4th Runge-Kutta method. In other words, the presented method is more accurate compared with Runge-Kutta technique.

5. CONCLUSIONS

A new numerical technique for dynamic analysis of SDOF structures has been proposed in
this paper. This method is based on the simultaneous use of energy and force equilibrium equations and is called MEM. This method with simple mathematical relations is illustrated on some standard examples for both linear and nonlinear systems. In the nonlinear cases, several types of nonlinear terms such as material, geometry and damping nonlinearities can be easily considered and the results obtained by the MEM method are more accurate compared with the 4th order Runge-Kutta method. Furthermore, one of the advantages of MEM compared with others existing method e.g. Newmark and Wilson is its independency from choosing the analytical parameters such as $\gamma, \beta$ in Newmark, and $\theta$ in Wilson. To sum up, the method presented in this study is an effective numerical technique that can be used in practical dynamic analyses especially in nonlinear cases without any essential limitation.

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