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FREE VIBRATION ANALYSIS OF RECTANGULAR MEMBRANES WITH VARIABLE DENSITY USING THE DISCRETE SINGULAR CONVOLUTION APPROACH

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ABSTRACT

Membranes are widely used in various engineering applications such as the design stage of microphones, pumps, pressure regulators, and other acoustical applications. This paper investigates the numerical aspects for free vibration analysis of homogeneous and non-homogeneous rectangular and square membranes. The method of discrete singular convolution is employed. The results are obtained for different density case and aspect ratios. Numerical results are presented and compared with that available in the literature. The results show that the regularized Shannon delta kernel based discrete singular convolution algorithm produces accurate frequency values.

Keywords: Frequency; discrete singular convolution; rectangular membrane; non-homogeneous density

1. INTRODUCTION

The method of discrete singular convolution [1] has been used recently for the vibration analysis of structures. Discrete singular convolution (DSC) method has emerged as a new approach for numerical solutions of differential equations. This new method has a potential approach for computer realization as a wavelet collocation scheme [2-4]. The use of the discrete singular convolution method for vibration analysis of beams, plates and shells [5-12] have been proven to be quite satisfactory. Free vibration analysis of plates and shells has also been investigated by the present author [13-19]. Membranes are widely used in various engineering applications. Therefore, free vibration analysis of such structures is a most important task for engineer in the design stage of microphones, pumps, pressure regulators, and other acoustical applications. In the literature, various methods have been used for vibration analysis of rectangular and circular membranes. Free vibration problem of

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membrane was considered by Wei [2] using the method of DSC. The method of differential quadrature was applied for frequency analysis of rectangular and circular membranes by Laura et al. [35,36]. Ho and Chen [32] applied a hybrid method for vibration of non-homogeneous membranes. Free vibration analysis of arbitrarily shaped membranes have been investigated by Houmat [28,29] using the Fourier *p*-element finite element. Some exact solutions of the eigenvalue problems of membranes have been reported by Kang et al. [24] and Kang and Lee [25,34] using non-dimensional dynamic influence function. Wu et al. [26] used radial basis function-based differential quadrature method to obtain the frequencies and mode shapes of arbitrary shaped membrane. Some important studies concerning analysis of membranes have been carried out, namely Jabareen and Eisenberger [33], Masad [30], Pronsato et al. [37], Leung et al. [27], Buchanan and Peddieson [20,21], Filipich and Rosales [31] and Buchanan [22]. Free vibration of circular and rectangular membrane is investigated by differential quadrature and DSC method [38]. The aim of the present paper is to present the DSC method for free vibration analysis of non-homogeneous rectangular membranes. This is the first instance in which the DSC method has been adopted for free vibration analysis of rectangular membranes with variable density.

2. DISCRETE SINGULAR CONVOLUTION (DSC)

The discrete singular convolution (DSC) method is an efficient and useful approach for the numerical solutions of differential equations [1]. This method introduced by Wei [1,2]. Like some other numerical methods, the DSC method discretizes the spatial derivatives and, therefore, reduces the given partial differential equations into an eigenvalue problem. The mathematical foundation of the DSC algorithm is the theory of distributions and wavelet analysis. Consider a distribution, T and $\eta(t)$ as an element of the space of the test function. A singular convolution can be defined by [3]

$$F(t) = (T * \eta)(t) = \int_{-\infty}^{\infty} T(t-x)\eta(x)dx \,, \tag{1}$$

where $T(t-x)$ is a singular kernel. For example, singular kernels of delta type [4]

$$T(x) = \delta^{(n)}(x)\,; \quad (n = 0,1,2,...,). \tag{2}$$

Kernel $T(x) = \delta(x)$ is important for interpolation of surfaces and curves, and $T(x) = \delta^{(n)}(x)$ for n>1 are essential for numerically solving differential equations. With a sufficiently smooth approximation, it is more effective to consider a discrete singular convolution [6]

$$F_\alpha(t) = \sum_k T_\alpha(t - x_k) f(x_k) \,, \tag{3}$$

where F_α (t) is an approximation to $F(t)$ and $\{x_k\}$ is an appropriate set of discrete points on which the DSC (3) is well defined. Note that, the original test function $\eta(x)$ has been replaced by f(x). This new discrete expression is suitable for computer realization. Recently, the use of some new kernels and regularizer such as delta regularizer was proposed to solve applied mechanics problem. The Shannon's kernel is regularized as [7]

$$\delta_{\Delta,\sigma}(x-x_k)=\frac{\sin[(\pi/\Delta)(x-x_k)]}{(\pi/\Delta)(x-x_k)}\exp\left[-\frac{(x-x_k)^2}{2\sigma^2}\right];\ \sigma>0. \tag{4}$$

where Δ is the grid spacing. It is also known that the truncation error is very small due to the use of the Gaussian regularizer, the above formulation given by Eq. (4) is practically and has an essentially compact support for numerical interpolation. In the DSC method, the function $f(x)$ and its derivatives with respect to the x coordinate at a grid point x_i are approximated by a linear sum of discrete values $f(x_k)$ given by [8]

$$\left.\frac{d^n f(x)}{dx^n}\right|_{x=x_i}=f^{(n)}(x)\approx\sum_{k=-M}^{M}\delta^{(n)}_{\Delta,\sigma}(x_i-x_k)f(x_k);\ (n=0,1,2,...,). \tag{5}$$

where $\delta_\Delta(x-x_k)=\Delta\delta_\alpha(x-x_k)$ and superscript (n) denotes the n*th*-order derivative, and $2M+1$ is the computational bandwidth which is centered around x and is usually smaller than the whole computational domain. For example the second order derivative at $x=x_i$ of the DSC kernels for directly given [8]

$$f^{(2)}(x)=\left.\frac{d^2 f}{dx^2}\right|_{x=x_i}\approx\sum_{k=-M}^{M}\delta^{(2)}_{\Delta,\sigma}(k\Delta x_N)f_{i+k,j}. \tag{6}$$

Second-order derivative in Eq. (6) is given as [3]

$$\begin{aligned}\delta^{(2)}_{\pi/\Delta,\sigma}(x_m-x_k)=&-\frac{(\pi/\Delta)\sin(\pi/\Delta)(x-x_k)}{(x-x_k)}\exp[-(x-x_k)^2/2\sigma^2]\\&-2\frac{\cos(\pi/\Delta)(x-x_k)}{(x-x_k)^2}\exp[-(x-x_k)^2/2\sigma^2)]\\&-2\frac{\cos(\pi/\Delta)(x-x_k)}{\sigma^2}\exp[-(x-x_k)^2/2\sigma^2]+2\frac{\sin(\pi/\Delta)(x-x_k)}{\pi(x-x_k)^3/\Delta}\exp[-(x-x_k)^2/2\sigma^2)]\\&+\frac{\sin(\pi/\Delta)(x-x_k)}{\pi(x-x_k)\sigma^2/\Delta}\exp[-(x-x_k)^2/2\sigma^2]+\frac{\sin(\pi/\Delta)(x-x_k)}{\pi\sigma^4/\Delta}(x-x_k)\exp[-(x-x_k)^2/2\sigma^2)]\end{aligned}$$

For $x=x_k$, this derivative is given by

$$\delta^{(2)}_{\sigma,\Delta}(0) = -\frac{3+(\pi^2/\Delta^2)\sigma^2}{3\sigma^2} = -\frac{1}{\sigma^2} - \frac{\pi^2}{3\Delta^2} \tag{7}$$

3. PROBLEM FORMULATIONS AND SOLUTION

As shown in Figure 1, consider a free vibration problem of pre-stretched non-homogeneous rectangular membrane. The non-dimensional form of governing equation is written as follows

$$\frac{\partial^2 W(x,y)}{\partial X^2} + \lambda^2 \frac{\partial^2 W(x,y)}{\partial Y^2} + \rho(x)\Omega^2 W(x,y) = 0, \tag{8}$$

Where W is the transverse deflection, ρ is the mass per unit area, ω is the circular frequency, and T is the tension per unit length.

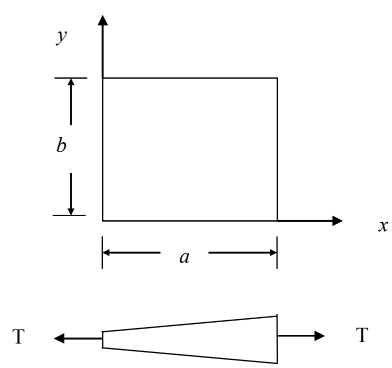


Figure 1. Non-homogenous rectangular membrane

In Eq. (8) the non-dimensional variables have been used given below

$$X = x/a, Y = y/b,\ \Omega^2 = \rho_0\omega^2 a^2/T\ ,\ \lambda = a/b\ . \tag{9}$$

In the present study, the density of membrane is a linear function of the variable x. The density is defined by

$$\rho x) = \rho_0[1+\alpha(x/a)] \tag{10}$$

Applying the discrete singular convolution to the governing equation yields

$$\sum_{k=-M}^{M} \delta^{(2)}_{\Delta,\sigma}(k\Delta x) W_{i+k,j} + \lambda^2 \sum_{k=-M}^{M} \delta^{(2)}_{\Delta,\sigma}(k\Delta y) W_{i,j+k} + \rho(x)\Omega^2 W_{ij} = 0, \tag{11}$$

The boundary conditions are as follows:

$$W=0 \text{ at edges} \tag{12}$$

4. NUMERICAL RESULTS

Several numerical examples are presented to demonstrate the accuracy of the present method. The homogenous case of a rectangular membrane is considered first. The obtained frequencies values are shown in Table 1. In this table the convergence characteristics of the fundamental frequency parameters of homogeneous rectangular membranes are depicted. The results are presented for different aspect ratios. As can be seen, the present DSC results compare very well with the analytical solutions and differential quadrature solutions given by Laura et al. [35]. It is also shown that, the sufficient accuracy has been obtained for N=15. The results for the first nine frequency values are listed in Table 2 along with analytical solutions. The results are presented for different values of α and λ parameters. The results of this study are similar to Laura's results [35]; the frequency value decreases with the increasing of aspect ratio.

Table 1. Comparison of fundamental frequencies of homogeneous rectangular membranes

λ	DQ Method Laura et al. [35]	Galerkin Method Laura et al. [35]	Present DSC results		
			***N*=11**	***N*=15**	***N*=17**
0.4	8.45901	8.45901	8.45906	8.45901	8.45901
0.6	6.10617	6.10618	6.10620	6.10618	6.10617
1.0	4.44289	4.44291	4.44305	4.44290	4.44290

Table 2. Comparison of fundamental frequencies of non- homogeneous membranes

λ	Laura et al. [35]		Present DSC results		
	α=0.1	α=1	α=0.1	α=0.5	α=1
0.2	15.61334	12.56414	15.61335	13.2865	12.56417
0.4	8.25221	6.79752	8.25222	7.4682	6.79753
0.6	5.95790	4.94214	5.95790	5.2374	4.94216
0.8	4.90719	4.08144	4.90722	4.4381	4.08145
1	4.33539	3.61043	4.33539	4.0036	3.61043

Table 3. First six frequencies of square membrane

N	**Mode number**					
	1	**2**	**3**	**4**	**5**	**6**
7	4.4536	7.0997	8.9685			
9	4.4439	7.0441	8.9362	9.9587		
11	4.4430	7.0255	8.8917	9.9358	11.3356	
13	4.4431	7.0251	8.8861	9.9355	11.3279	12.9605
15	4.4429	7.0249	8.8858	9.9349	11.3276	12.9536
17	4.4429	7.0249	8.8858	9.9347	11.3273	12.9533
19	4.4429	7.0249	8.8858	9.9347	11.3273	12.9532

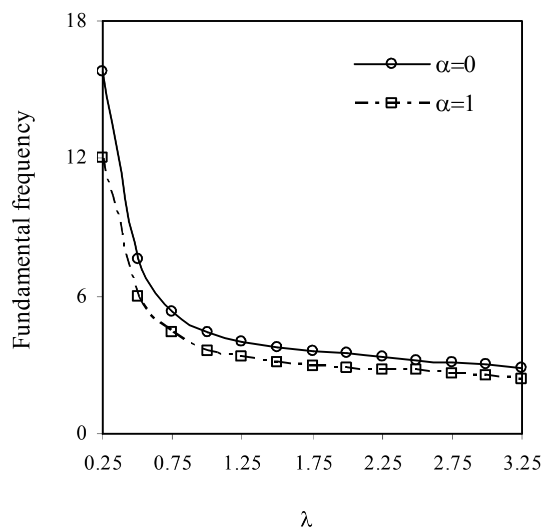


Figure 2. Variation of fundamental frequency with aspect ratio

The effect of taper ratio on the fundamental frequency of vibration is shown in Fig. 2 for two different values of α . It can be seen that the fundamental frequency decreases with the aspect ratio at various α. The first six frequencies are listed in Table 3 for different grid numbers. It is found that very good convergence is obtained with increasing of grid numbers. It is seen from this table that when the grid point numbers reaches N=15 the present method gives accurate predictions for the first four vibration modes. For higher modes of vibration, however, the accurate results are obtained for N=17.

Table 4. Comparison of frequency values of the rectangular membrane (b=1.2; a=0.9)

Mode number	**Methods**			
	Exact Kang et al. [24]	**DQ Wu et al. [26]**	**FEM Kang et al. [24]**	**Present DSC**
1	4.3633	4.3633	4.3651	4.3633
2	6.2929	6.2929	6.3006	6.2930
3	7.4560	7.4561	7.4669	7.4561
4	8.5947	8.5948	8.6213	8.5947
5	8.7266	8.7267	8.7407	8.7267
6	10.5083	10.5083	10.5370	10.5085

Table 5. Frequency values of the rectangular membrane for different mode shapes (λ=0.6; b=1)

n	**m**					
	1	**2**	**3**	**4**	**5**	**6**
1	6.10617	10.93102	16.01603	21.17428	26.36285	31.56684
2	8.17683	12.21004	16.91481	21.86204	26.91839	32.03209
3	10.7794	14.08617	18.31502	22.96251	27.81951	32.79317
4	13.6115	16.35466	20.11236	24.42003	29.03426	33.82966
5	16.5538	18.87506	22.21024	26.17507	30.52518	35.11748
6	19.5594	21.55903	24.53128	28.17191	32.25382	36.63005

Frequency values of the rectangular membrane (λ=0.6) for different mode shapes are listed in Table 4 and Table 5. Frequency values obtained by DSC method are presented in Table 4 together with the finite element solutions [24], differential quadrature [26] and exact solution [24]. The DSC results are generally in agreement with the results produced from the analytical [24] and the FEM results [24]. It is seen in these two tables that the present method yields accurate results. The natural frequency parameters of the first ten modes of vibration are given in Figs. 4-5 for square membrane. The results have been presented for E=40 N/mm^2, $\rho=1\times10^{-6}$ kg/mm^3, h=1mm, a=100mm, υ=0.3. Two different tension values are chosen. It is shown that the increasing value of *T* always increases the frequency parameter. The comparisons presented in Tables 1-4 illustrate that the DSC algorithm is capable of yielding accurate results for the vibration analysis of membranes.

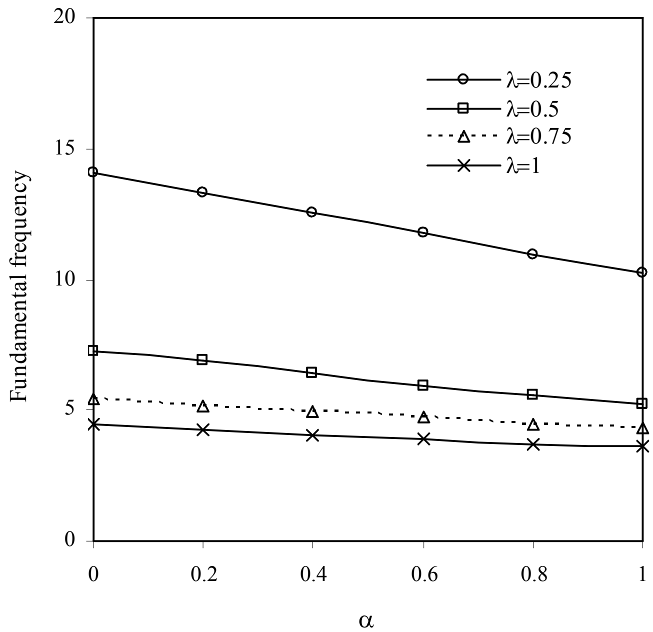
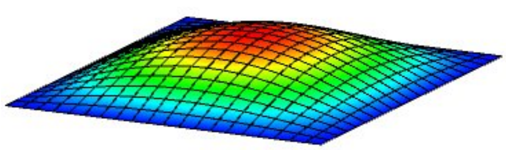
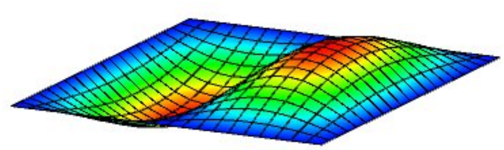


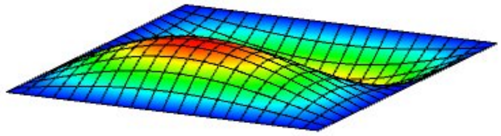
Figure 3. Variation of frequency with density for different taper ratio



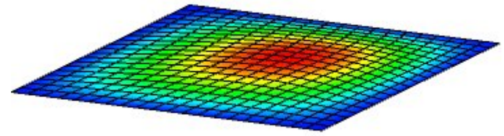
Mode 1: 22.384 Hz.



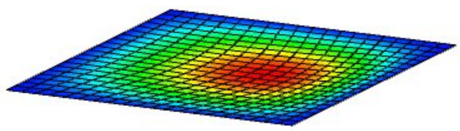
Mode 2: 35.479 Hz.



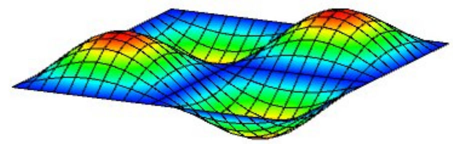
Mode 3: 35.478 Hz.



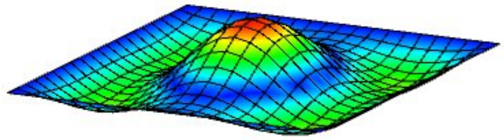
Mode 4: 43.971 Hz.



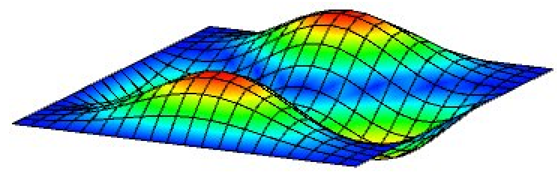
Mode 5: 43.971 Hz.



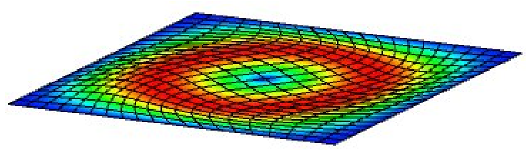
Mode 6: 44.905 Hz.



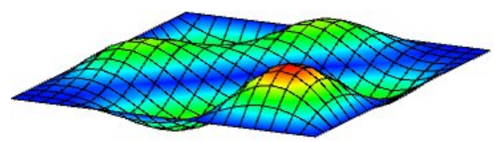
Mode 7: 50.423 Hz.



Mode 8: 50.423 Hz.

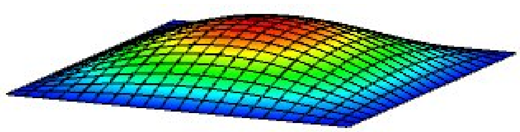


Mode 9: 57.393 Hz.

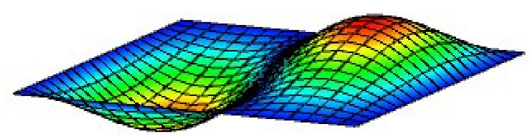


Mode 10: 57.447 Hz.

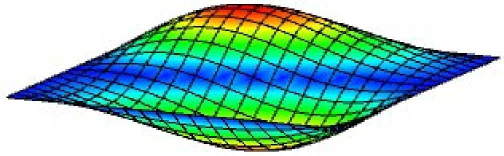
Figure 4. Frequency and mode shapes of square membrane (a=100 mm, T=10 N/mm, E=40 N/mm^2, $\rho=1\times10^{-6}$ kg/mm^3, h=1mm, υ=0.3)



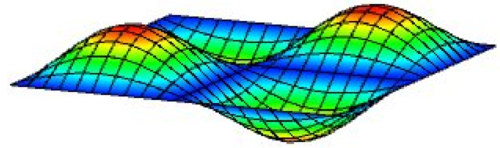
Mode 1: 7.078 Hz.



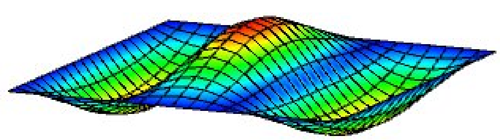
Mode 2: 11.219 Hz.



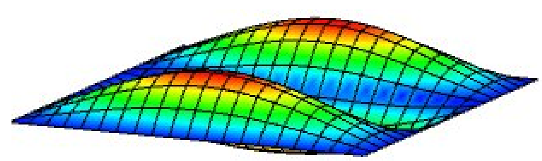
Mode 3: 11.219 Hz.



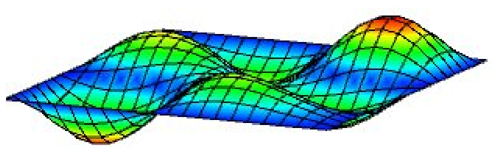
Mode 4: 14.200 Hz.



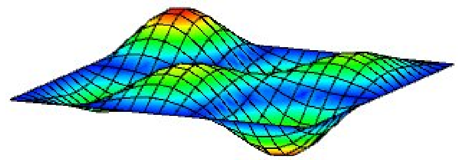
Mode 5: 15.945 Hz.



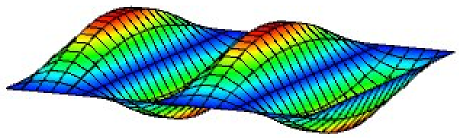
Mode 6: 15.945 Hz.



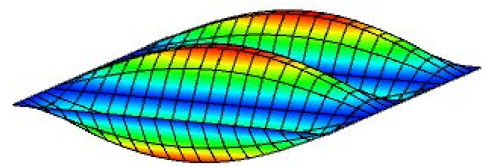
Mode 7: 18.166 Hz.



Mode 8: 18.166 Hz.



Mode 9: 20.937 Hz.



Mode 10: 20.937 Hz.

Figure 5. Frequency and mode shapes of square membrane (a=100 mm, T=1 N/mm, E=40 N/mm^2, $\rho=1\times10^{-6}$ kg/mm^3, h=1mm, $\upsilon=0.3$)

5. CONCLUSIONS

In this paper, discrete singular convolution has been proposed for the free vibration analysis of rectangular membrane with variable density. Several numerical examples confirmed the validity and effectiveness of the present numerical approach. It was seen from the numerical examples that the proposed algorithm always gives the convergence as the number of grid points increases. The present method can be further developed for analyzing membranes having arbitrary straight sided quadrilateral or curvilinear domain.

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