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ASSESSMENT OF LEAST-SQUARES FINITE ELEMENT MODELS OF BEAMS

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ABSTRACT

The purpose of this study is to investigate the effectiveness of the least-squares based finite element models in solving the beam bending problems to overcome shear and membrane locking and predict generalized forces accurately. This study is conducted using the Euler-Bernoulli and Timoshenko beam theories applied to straight beams. The solution accuracy of the least-squares finite element models with conventional finite element models is also assessed.

Keywords: Beams; Euler-Bernoulli beam theory; finite element model; least-squares finite element models; Timoshenko beam theory

1. INTRODUCTION

There are some numerical challenges that are encountered with conventional finite element models based on the weak form Galerkin formulation, which is the most common in practice [1,2]. In these models, the secondary variables such as the bending moment and shear force are post-computed, typically at Gauss points and not at the nodes, and do not yield good accuracy. In addition, in the case of the Timoshenko beam theory, the element with lower-order equal interpolation of the generalized displacements suffers from shear locking. In both Euler-Bernoulli and Timoshenko beam theories, the elements based on the weak form Galerkin formulation also suffer from membrane locking [1,2] when applied to geometrically nonlinear problems. Both types of locking are a result of using inconsistent interpolation for the variables involved in the formulation. In order to alleviate these types of locking, often reduced integration techniques are employed. However, such ad-hoc techniques have other disadvantages, such as hour-glass modes or spurious rigid body modes. Thus, it is desirable to develop alternative finite element models that overcome the locking problems and yield good accuracy for stress resultants. Least-squares finite element models are considered to be alternatives to the weak form Galerkin finite element model

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and thus considered in this study for investigation. The least-squares formulation helps to retain the generalized displacements and forces (or stress resultants) as independent variables, and also allows the use of equal order interpolation functions for all variables.

The objective of this paper is to study the effectiveness of the least-squares finite element models compared to the weak form Galerkin finite element models of beams to overcome shear and membrane locking and predict generalized forces accurately. The Euler-Bernoulli and Timoshenko beam theories are used in this study. To achieve the objective, different finite element models of the two beam theories are developed and are applied to beam problems with different boundary conditions. Solutions obtained using the least-squares finite element models are compared to the solutions obtained from the conventional, weak form Galerkin finite element models. The following discussion provides the background for the present study.

Depending on the kinematic assumptions, two different theories are often used to model the structural behavior of beams: (1) Euler- Bernoulli beam theory (EBT) and (2) Timoshenko beam theory (TBT). In the Euler Bernoulli beam theory, one neglects the effect of the transverse shear strain whereas in the Timoshenko beam theory it is taken into account.

Both shear and membrane locking in beams are primarily due to the use of inconsistent interpolation of the variables. When equal and lower order interpolation of the displacement and rotation are used in the Timoshenko beam finite element, the element exhibits locking as it is unable to cope with the constraint that the slope should be compatible with the derivative of the deflection in the thin beam limit. The problem of shear locking is often overcome by numerically mimicking different variation (i.e., constant and linear) of the rotation function in shear energy and bending energy through numerical integration [2]. There are several other approaches that have been adopted to eliminate locking [1, 2, 3-9]. The concept of locking was first discussed by Kikuchi and Aizawa [3], and Zienkiewicz and Owen [10] advocated that the reduced integration technique is a means of obtaining accurate solutions. However, such ad-hoc approaches have other disadvantages, such as appearance of hour-glass modes or spurious rigid body modes. Hence, it is desirable to develop alternative finite element models that overcome the locking problems.

In the past few years finite element models based on least-squares variational principles have drawn considerable attention. Given a set of differential equations, the least-squares method allows one to define a convex, unconstrained minimization principle so that the finite element model can be developed in Ritz or weak form Galerkin setting [2, 11]. This model has proved to result in a positive-definite system of equations and significant savings in the computational cost [11].

The least-square approach has been implemented in the finite element context to solve the problems of plate bending, shear-deformable shells, incompressible and compressible fluid flows, fracture mechanics, and so on (see [11-22], among others). However, there has been no systematic study involving the development of least-squares finite element models of beam theories and their assessment in comparison to the conventional beam finite elements. The present study was undertaken to fill this gap in the literature. The present study also accounts for geometric nonlinearity in the von Karaman sense.

2. GOVERNING EQUATIONS OF EBT AND TBT

In the Euler Bernoulli beam theory is based on the assumption of that the plane cross sections perpendicular to the beam axis before deformation remain (a) plane (b) inextensible, and (c) rotate such that they remain perpendicular to the beam axis after deformation. These assumptions amount to the neglect of the Poisson effect and the transverse shear strain. In the Timoshenko beam theory, the last assumption, namely, the normality condition is not used and therefore transverse shear strain is accounted in a rudimentary way.

The displacement fields in the two theories are taken, consistent with the assumptions made, as [2]

$$\text{EBT:} \quad u_1 = u(x) - z\frac{dw}{dx}, \quad u_2 = 0, \quad u_3 = w(x) \tag{1}$$

$$\text{TBT:} \quad u_1 = u(x) + z\phi(x), \ u_2 = 0, \ u_3 = w(x) \tag{2}$$

where (u_1, u_2, u_3) are the displacement along (*x, y, z*) axes and u is the axial displacement of a point on the neutral axis, w is the transverse displacement of the point on the neutral axis of the beam, and ϕ is the rotation of transverse normal to the beam axis. The the von Karman nonlinear strains for the two theories and the equilibrium equations (the same for both theories) are presented below (see [2] for details).

Euler-Bernoulli Beam Theory

$$\varepsilon_{xx} = \varepsilon_{xx}^0 + z\varepsilon_{xx}^1, \ \varepsilon_{xx}^0 = \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2, \ \varepsilon_{xx}^1 = -\frac{d^2w}{dx^2} \tag{3}$$

Timoshenko Beam Theory

$$\varepsilon_{xx} = \varepsilon_{xx}^0 + z\varepsilon_{xx}^1, \ \varepsilon_{xx}^0 = \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2, \ \varepsilon_{xx}^1 = \frac{d\phi}{dx} \tag{4}$$

Equilibrium Equations

$$\begin{gathered} -\frac{dN}{dx} = f(x) \\ -\frac{d}{dx}\left(\frac{dw}{dx}N\right) - \frac{dV}{dx} = q(x) \\ V - \frac{dM}{dx} = 0 \end{gathered} \tag{5}$$

where N is the axial force, M is the bending moment, and Q is the transverse shear force

$$N = \int_{A^e} \sigma_{xx} dA, \ M = \int_{A^e} \sigma_{xx} z dA, \ Q = K_s \int_A \sigma_{xz} dA \tag{6}$$

K_s being the shear correction coefficient. The beam constitutive equations $(\sigma_{xx} = E\varepsilon_{xx}$ and $\sigma_{xx} = 2G\varepsilon_{xz})$ in the two theories are given as follows.

Euler-Bernoulli Beam Theory

$$N = EA\frac{du}{dx},\ M = -EI\frac{d^2w}{dx^2},\ V = -\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right) \tag{7}$$

Timoshenko Beam Theory

$$N = EA\frac{du}{dx},\ M = EI\frac{d\phi}{dx},\ V = GAK_s\left(\phi + \frac{dw}{dx}\right) \tag{8}$$

Here A denotes area of cross section, I the second moment of area, E the modulus of elasticity, and G is the shear modulus.

3. LEAST-SQUARES FINITE ELEMENT MODELS OF EBT AND TBT

3.1 Introduction

The displacement finite element models of EBT and TBT can be found in the book by Reddy [2]. Shear locking (occurs only in TBT) and membrane locking (occurs in both theories) are also discussed there. In order to avoid the two types of locking, different methods such as reduced integration method have been implemented in the past. But this approach also has its disadvantages of bringing hour-glass modes or spurious rigid body modes into the models. Thus, it is desirable to develop alternative finite element models that overcome the locking problems. An effort has been made to develop models that can use higher-order approximation functions, and finite element models were developed using the least-squares method. These models are discussed in the next section.

3.2 The Basic Idea of the Least-Squares Method

The basic idea behind least-squares models is to compute the residuals due to the approximation of the variables of each equation being modeled, construct integral statement of the sum of the squares of the residuals (called least-squares functional), and minimize the integral with respect to the unknown parameters of the approximations. To be more explicit, consider an operator equation of the form

$$A(u) = f \quad \text{in} \quad \Omega \text{ and} \quad \mathrm{B}(u) = g \quad \text{in } \Gamma \tag{9}$$

We seek suitable approximation of u as $u_h = \sum_{j=1}^{n} c_j\varphi_j$. In the least squares method, we seek the minimum of the sum of squares of the residuals in the approximation of equations as follows

$$\frac{\partial}{\partial c_i}\int_\Omega R^2\left(\mathrm{x}, c_j\right)dx = 0$$

where

$$R^2 = R_1^2 + R_2^2,\ R_1 = A(u_h) - f,\ R_2 = B(u_h) - g$$

The necessary condition for the minimum is

$$0 = \delta I\left(u_h\right) = \delta\left\{\int_\Omega\left[A\left(u\right) - f\right]^2 dx + \oint_\Gamma\left[B\left(u\right) - g\right]^2 ds\right\} \tag{10}$$

Thus the variational problem is to seek u_h such that $B\left(\delta u_h, u_h\right) = l(u_h)$ holds for all δu_h. where

$$B\left(\delta u_h, u_h\right) = \int_\Omega \delta\left[A\left(u_h\right)\right]A\left(u_h\right)d\mathrm{x} + \oint_\Gamma \delta\left[B\left(u_h\right)\right]B\left(u_h\right)ds$$

$$l(u_h) = \int_\Omega \delta\left[A\left(u_h\right)\right]f d\mathrm{x} + \oint_\Gamma \delta\left[B\left(u_h\right)\right]g ds$$

Using the above concept, the two least-squares finite element models of both the Euler-Bernoulli beam theory (EBT) and the Timoshenko beam theory (TBT) are developed.

3.3 Least-Squares Finite Element Model of the EBT (EBT-1)

Consider the following governing equations

$$\begin{gathered} -\frac{dN}{dx} = f \\ -\frac{d^2M}{dx^2} - \frac{d}{dx}\left(N\frac{dw}{dx}\right) = q \\ M + EI\frac{d^2w}{dx^2} = 0 \end{gathered} \tag{11}$$

where N is known in terms of u and as

$$N = EA\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] \tag{12}$$

The linearization of the above equations that are used here are

$$\begin{gathered} -EA\left(\frac{d^2u}{dx^2} + \frac{d\overline{w}}{dx}\frac{d^2w}{dx^2}\right) = f \\ -\frac{d^2M}{dx^2} - EA\left(\frac{d^2u}{dx^2} + \frac{d\overline{w}}{dx}\frac{d^2w}{dx^2}\right)\frac{d\overline{w}}{dx} - \overline{N}\frac{d^2w}{dx^2} = q \end{gathered} \tag{13}$$

$$M + EI\left(\frac{d^2 w}{dx^2}\right) = 0$$

where $\bar{N} = EA\left(\frac{d\bar{u}}{dx} + \frac{1}{2}\left(\frac{d^2\bar{w}}{dx^2}\right)^2\right)$. The least-squares functional associated with the above set of linearized equations over a typical element occupying the domain (x_a, x_b) is

$$J_L(u_h, w_h, M_h) = \int_{x_b}^{x_a}\left[p_1\left[\frac{d^2 M_h}{dx^2} + EA\left(\frac{d^2 u_h}{dx^2} + \frac{d\bar{w}_h}{dx}\frac{d^2 w_h}{dx^2}\right)\frac{d\bar{w}_h}{dx} + \bar{N}\frac{d^2 w_h}{dx^2} + q\right]^2 + \right.$$
$$\left.\left[EA\left(\frac{d^2 u_h}{dx^2} + \frac{d\bar{w}_h}{dx}\frac{d^2 w_h}{dx^2}\right) + f\right]^2 + p_2\left(M_h + EI\frac{d^2 w_h}{dx^2}\right)^2\right] \tag{14}$$

where p_1 and p_2 are scaling factors to make the entire residual to have the same physical dimensions, and quantities with bar are assumed to be known from the previous iteration and hence their variations are zero.

The necessary condition for the minimum of J_L is $\delta J_L = 0$

$$0 = \int_{x_b}^{x_a}\left[p_1\left[\frac{d^2 M_h}{dx^2} + EA\left(\frac{d^2 u_h}{dx^2} + \frac{d\bar{w}_h}{dx}\frac{d^2 w_h}{dx^2}\right)\frac{d\bar{w}_h}{dx} + \bar{N}\frac{d^2 w_h}{dx^2} + q\right]\right.$$
$$\times\left[\frac{d^2 \delta M_h}{dx^2} + EA\left(\frac{d^2 \delta u_h}{dx^2} + \frac{d\bar{w}_h}{dx}\frac{d^2 \delta w_h}{dx^2}\right)\frac{d\bar{w}_h}{dx} + \bar{N}\frac{d^2 \delta w_h}{dx^2}\right] +$$
$$EA\left[EA\left(\frac{d^2 u_h}{dx^2} + \frac{d\bar{w}_h}{dx}\frac{d^2 w_h}{dx^2}\right) + f\right]\left(\frac{d^2 \delta u_h}{dx^2} + \frac{d\bar{w}_h}{dx}\frac{d^2 \delta w_h}{dx^2}\right) + \tag{15}$$
$$\left. p_2\left(\delta M_h + EI\frac{d^2 \delta w_h}{dx^2}\right)\left(M_h + EI\frac{d^2 w_h}{dx^2}\right)\right] dx$$

The above statement is equivalent to the following three integral statements:

$$0 = \int_{x_b}^{x_a}\left[(EA)^2\frac{d^2\delta u}{dx^2}\frac{d^2 u}{dx^2} + p_1 (EA)^2\left(\frac{d\bar{w}}{dx}\right)^2\frac{d^2\delta u}{dx^2}\frac{d^2 u}{dx^2} + (EA)^2\frac{d\bar{w}}{dx}\frac{d^2\delta u}{dx^2}\frac{d^2 w}{dx^2} + \right.$$
$$\left. p_1 (EA)^2\frac{d\bar{w}}{dx}\hat{N}\frac{d^2\delta u}{dx^2}\frac{d^2 w}{dx^2} + p_1 EA\frac{d\bar{w}}{dx}\frac{d^2\delta u}{dx^2}\frac{d^2 M}{dx^2} + EA\frac{d^2\delta u}{dx^2}\left(f + p_1\frac{d\bar{w}}{dx}q\right)\right] dx \tag{16}$$

$$0=\int_{x_b}^{x_a}\Bigg[(EA)^2\frac{d\bar{w}}{dx}\frac{d^2\delta w}{dx^2}\frac{d^2u}{dx^2}+(EA)^2\,p_1\hat{N}\frac{d\bar{w}}{dx}\frac{d^2\delta w}{dx^2}\frac{d^2u}{dx^2}+p_2\,(EI)^2\frac{d^2\delta w}{dx^2}\frac{d^2w}{dx^2}+$$
$$(EA)^2\left(\frac{d\bar{w}}{dx}\right)^2\frac{d^2\delta w}{dx^2}\frac{d^2w}{dx^2}+(EA)^2\,p_1\hat{N}^2\frac{d^2\delta w}{dx^2}\frac{d^2w}{dx^2}+p_2EI\frac{d^2\delta w}{dx^2}M+ \tag{17}$$
$$EAp_1\hat{N}\frac{d^2\delta w}{dx^2}\frac{d^2M}{dx^2}+\left(EA\frac{d\bar{w}}{dx}f+EAp_1\hat{N}q\right)\frac{d^2\delta w}{dx^2}\Bigg]dx$$

$$0=\int_{x_b}^{x_a}\Bigg[EAp_1\frac{d\bar{w}}{dx}\frac{d^2\delta M}{dx^2}\frac{d^2u}{dx^2}+p_2\delta MEI\frac{d^2w}{dx^2}+p_1EA\hat{N}\frac{d^2\delta M}{dx^2}\frac{d^2w}{dx^2}+p_2M\delta M$$
$$+p_1\frac{d^2\delta M}{dx^2}\frac{d^2M}{dx^2}+\frac{d^2\delta M}{dx^2}p_1q\Bigg]dx \tag{18}$$

where

$$N=\left[\frac{d\bar{u}}{dx}+\frac{1}{2}\left(\frac{d\bar{w}}{dx}\right)^2\right],\ \ \hat{N}=\bar{N}+\left(\frac{d\bar{w}}{dx}\right)^2=\left[\frac{d\bar{u}}{dx}+\frac{3}{2}\left(\frac{d\bar{w}}{dx}\right)^2\right] \tag{19}$$

Since the physics of the Euler Bernoulli's Beam theory requires the specification of $u, w, \theta=-\frac{dw}{dx}, N, M$ and $V=\frac{dM}{dx}$ we seek Hermite cubic approximations of u_h, w_h and M_h

$$u_h=\sum_{j=1}^{4}\Delta_j^1\varphi_j(x),\ \ w_h=\sum_{j=1}^{4}\Delta_j^2\varphi_j(x)\ \text{and}\ M_h=\sum_{j=1}^{4}\Delta_j^3\varphi_j(x) \tag{20}$$

where Δ_j^1, Δ_j^2 and Δ_j^3 denote the nodal values of $\left(u_h,-\frac{du_h}{dx}\right),\left(w_h,-\frac{dw_h}{dx}\right)$ and $\left(M_h,-\frac{dM_h}{dx}\right)$ respectively at the jth node and $\varphi_j(x)$ are the Hermite cubic interpolation functions. Substituting the above equations into the statements (16)-(18), we obtain the finite element equations

$$\begin{bmatrix}[K^{11}] & [K^{12}] & [K^{13}]\\ [K^{21}] & [K^{22}] & [K^{23}]\\ [K^{31}] & [K^{32}] & [K^{33}]\end{bmatrix}\begin{Bmatrix}\{\Delta^1\}\\ \{\Delta^2\}\\ \{\Delta^3\}\end{Bmatrix}=\begin{Bmatrix}\{F^1\}\\ \{F^2\}\\ \{F^3\}\end{Bmatrix} \tag{21}$$

where

$$K_{ij}^{11}=\int_{x_b}^{x_a}(EA)^2\frac{d^2\varphi_i}{dx^2}\frac{d^2\varphi_j}{dx^2}dx+p_1\int_{x_b}^{x_a}(EA)^2\left(\frac{d\bar{w}}{dx}\right)^2\frac{d^2\varphi_i}{dx^2}\frac{d^2\varphi_j}{dx^2}dx$$

$$K_{ij}^{12} = \int_{x_b}^{x_a} \left(EA\right)^2 \left(1 + p_1 \hat{N}\right) \frac{d\overline{w}}{dx} \frac{d^2\varphi_i}{dx^2} \frac{d^2\varphi_j}{dx^2} dx$$

$$K_{ij}^{13} = p_1 \int_{x_b}^{x_a} EA \frac{d\overline{w}}{dx} \frac{d^2\varphi_i}{dx^2} \frac{d^2\varphi_j}{dx^2} dx$$

$$K_{ij}^{21} = \int_{x_b}^{x_a} \left(EA\right)^2 \left(1 + p_1 \hat{N}\right) \frac{d\overline{w}}{dx} \frac{d^2\varphi_i}{dx^2} \frac{d^2\varphi_j}{dx^2} dx$$

$$K_{ij}^{22} = \int_{x_b}^{x_a} \left(EA\right)^2 \left[\left(\frac{d\overline{w}}{dx} \right)^2 + p_1 \hat{N}^2 \right] \frac{d^2\varphi_i}{dx^2} \frac{d^2\varphi_j}{dx^2} dx + p_2 \left(EI\right)^2 \int_{x_b}^{x_a} \frac{d^2\varphi_i}{dx^2} \frac{d^2\varphi_j}{dx^2} dx$$

$$K_{ij}^{23} = \int_{x_b}^{x_a} p_1 EA\hat{N} \frac{d^2\varphi_i}{dx^2} \frac{d^2\varphi_j}{dx^2} dx + p_2 EI \int_{x_b}^{x_a} \frac{d^2\varphi_i}{dx^2} \varphi_j dx$$

$$K_{ij}^{31} = \int_{x_b}^{x_a} p_1 EA \frac{d\overline{w}}{dx} \frac{d^2\varphi_i}{dx^2} \frac{d^2\varphi_j}{dx^2} dx$$

$$K_{ij}^{32} = \int_{x_b}^{x_a} p_1 \left(EA\right)^2 \hat{N} \frac{d^2\varphi_i}{dx^2} \frac{d^2\varphi_j}{dx^2} dx + p_2 EI \int_{x_b}^{x_a} \frac{d^2\varphi_i}{dx^2} \varphi_j dx$$

$$K_{ij}^{33} = \int_{x_b}^{x_a} p_1 \frac{d^2\varphi_i}{dx^2} \frac{d^2\varphi_j}{dx^2} dx + p_2 \int_{x_b}^{x_a} \varphi_i \varphi_j dx$$

$$F_i^1 = -EA \int_{x_b}^{x_a} \left[f \frac{d^2\varphi_i}{dx^2} + q p_1 \frac{d\overline{w}}{dx} \frac{d^2\varphi_i}{dx^2} \right] dx$$

$$F_i^2 = -\int_{x_b}^{x_a} \left[EAf \frac{d\overline{w}}{dx} + q p_1 EA\hat{N} \right] \frac{d^2\varphi_i}{dx^2} dx$$

$$F_i^3 = -p_1 \int_{x_b}^{x_a} q \frac{d^2\varphi_i}{dx^2} dx \tag{22}$$

From the terms of K_{ij}^{33} it is clear that the terms p_1 and p_2 should be taken such that $p_2 = p_1 / h^2$, where h is the element length.

3.4 Least-Squares Finite Element of the TBT (TBT-1)

The least-squares finite element model of the following set of nonlinear equations of TBT, assuming EA , EI, GAK_s as constant, was developed:

$$-\frac{dN}{dx} = f$$

$$-\frac{d}{dx}\left[GAK_S \left(\phi + \frac{dw}{dx} \right) \right] - \frac{d}{dx}\left(N \frac{dw}{dx} \right) = q \tag{23}$$

$$-\frac{d}{dx}\left(EI\,\frac{d\phi}{dx}\right)+GAK_S\left(\phi+\frac{dw}{dx}\right)=0$$

The following linearization of the above equations is used:

$$\begin{gathered}-EA\left(\frac{d^2u}{dx^2}+\frac{d\bar{w}}{dx}\frac{d^2w}{dx^2}\right)=f\\ -GAK_S\left(\frac{d\phi}{dx}+\frac{d^2w}{dx^2}\right)-EA\frac{d^2u}{dx^2}\frac{d\bar{w}}{dx}-\bar{N}\,\frac{d^2w}{dx^2}=q\\ -\frac{d\left(EI\,\dfrac{d\phi}{dx}\right)}{dx}+GAK_S\left(\phi+\frac{dw}{dx}\right)=0\end{gathered} \tag{24}$$

where

$$\bar{N}=EA\left(\frac{d\bar{u}}{dx}+\frac{1}{2}\left(\frac{d^2\bar{w}}{dx^2}\right)^2\right),\ \hat{N}=EA\left(\frac{d\bar{u}}{dx}+\frac{3}{2}\left(\frac{d^2\bar{w}}{dx^2}\right)^2\right) \tag{25}$$

The least-squares functional associated with the above set of linearized equations over a typical element is

$$\begin{aligned}J_L(u_h,w_h,\phi_h)=\int_{x_b}^{x_a}\Bigg\{&p_1\left[-GAK_S\left(\frac{d\phi_h}{dx}+\frac{d^2w_h}{dx^2}\right)+EA\frac{d^2u_h}{dx^2}\frac{d\bar{w}_h}{dx}+\hat{N}\frac{d^2w_h}{dx^2}+q\right]^2+\\ &p_2\left[-EI\frac{d^2\phi_h}{dx^2}+GAK_S\left(\phi_h+\frac{dw_h}{dx}\right)\right]^2+\left[EA\left(\frac{d^2u_h}{dx^2}+\frac{d^2w_h}{dx^2}\frac{d\bar{w}_h}{dx}\right)+f\right]^2\Bigg\}dx\end{aligned} \tag{26}$$

where p_1 and p_2 are scaling factors to make the entire residual to have the same physical dimensions, and quantities with bar are assumed to be known from the previous iteration and their variations are zero.

The necessary condition for the minimum of J_L is δJ_L =0, which is equivalent to the following three statements

$$\begin{aligned}0=\int_{x_b}^{x_a}\Bigg[&\frac{d^2\delta u_h}{dx^2}EA\left(EA\frac{d^2u}{dx^2}+EA\frac{d\bar{w}}{dx}\frac{d^2w}{dx^2}+f\right)+\\ &p_1\frac{d\bar{w}}{dx}EA\frac{d^2\delta u}{dx^2}\left(-GAK_S\left(\frac{d\phi_h}{dx}+\frac{d^2w_h}{dx^2}\right)+EA\frac{d\bar{w}_h}{dx}\frac{d^2u_h}{dx^2}+\hat{N}\frac{d^2w_h}{dx^2}+q\right)\Bigg]dx\end{aligned} \tag{27}$$

$$0=\int_{x_b}^{x_a}\Bigg[p_2 GAK_S\frac{d^2\delta w_h}{dx^2}\left(-EI\frac{d^2\delta\phi_h}{dx^2}+GAK_S\left(\phi_h+\frac{dw_h}{dx}\right)\right)+ EA\frac{d\overline{w}_h}{dx}\frac{d^2\delta w_h}{dx^2}\left[EA\left(\frac{d^2u_h}{dx^2}+\frac{d\overline{w}_h}{dx}\frac{d^2w_h}{dx^2}\right)+f\right]+p_1\left(GAK_S\frac{d^2\delta w_h}{dx^2}+\hat{N}\frac{d^2\delta w_h}{dx^2}\right) \left(GAK_S\left(\frac{d\phi_h}{dx}+\frac{d^2w_h}{dx^2}\right)+EA\frac{d\overline{w}_h}{dx}\frac{d^2u_h}{dx^2}+\hat{N}\frac{d^2w_h}{dx^2}+q\right)\Bigg]dx \tag{28}$$

$$0=\int_{x_b}^{x_a}\Bigg[p_1-GAK_S\frac{d\delta\phi_h}{dx}\left(-GAK_S\left(\frac{d\phi_h}{dx}+\frac{d^2w_h}{dx^2}\right)+EA\frac{d\overline{w}_h}{dx}\frac{d^2u_h}{dx^2}+\hat{N}\frac{d^2w_h}{dx^2}+q\right)+ p_2\left(-EI\frac{d^2\delta\phi_h}{dx^2}+GAK_S\delta\phi_h\right)\left(GAK_S\left(\phi_h+\frac{dw_h}{dx}\right)-EI\frac{d^2\phi_h}{dx^2}\right)\Bigg]dx \tag{29}$$

where

$$N=\left[\frac{d\overline{u}}{dx}+\frac{1}{2}\left(\frac{d\overline{w}}{dx}\right)^2\right],\quad \hat{N}=\overline{N}+\left(\frac{d\overline{w}}{dx}\right)^2=\left[\frac{d\overline{u}}{dx}+\frac{3}{2}\left(\frac{d\overline{w}}{dx}\right)^2\right] \tag{30}$$

Since the physics of the Euler Bernoulli's Beam theory requires the specification of $u, w, \theta=-\frac{dw}{dx}, N, M$ and $V=\frac{dM}{dx}$ we seek Hermite cubic approximations of $u_h \,.\, w_h$ and M_h

$$u_h=\sum_{j=1}^{4}\Delta_j^1\varphi_j(x),\quad w_h=\sum_{j=1}^{4}\Delta_j^2\varphi_j(x) \text{ and } M_h=\sum_{j=1}^{4}\Delta_j^3\varphi_j(x) \tag{31}$$

where Δ_j^1, Δ_j^2 and Δ_j^3 denote the nodal values of $\left(u_h, -\frac{du_h}{dx}\right), \left(w_h, -\frac{dw_h}{dx}\right)$ and $\left(M_h, -\frac{dM_h}{dx}\right)$, respectively at the jth node and $\varphi_j(x)$ are the Hermite cubic interpolation functions. Substituting the approximations (31) into the integral statements (27)-(29), we obtain the finite element model:

$$\begin{bmatrix}[K^{11}] & [K^{12}] & [K^{13}]\\ [K^{21}] & [K^{22}] & [K^{23}]\\ [K^{31}] & [K^{32}] & [K^{33}]\end{bmatrix}\begin{Bmatrix}\{\Delta^1\}\\ \{\Delta^2\}\\ \{\Delta^3\}\end{Bmatrix}=\begin{Bmatrix}\{F^1\}\\ \{F^2\}\\ \{F^3\}\end{Bmatrix} \tag{32}$$

where

$$K_{ij}^{11} = \int_{x_b}^{x_a} (EA)^2 \frac{d^2\varphi_i}{dx^2}\frac{d^2\varphi_j}{dx^2}dx + p_1 \int_{x_b}^{x_a} (EA)^2 \left(\frac{d\overline{w}_h}{dx}\right)^2 \frac{d^2\varphi_i}{dx^2}\frac{d^2\varphi_j}{dx^2}dx$$

$$K_{ij}^{12} = \int_{x_b}^{x_a} (EA)\left(EA + p_1\hat{N} + p_1 GAK_S\right)\frac{d\overline{w}_h}{dx}\frac{d^2\varphi_i}{dx^2}\frac{d^2\varphi_j}{dx^2}dx$$

$$K_{ij}^{13} = p_1 GAK_S \int_{x_b}^{x_a} EA\frac{d\overline{w}_h}{dx}\frac{d^2\varphi_i}{dx^2}\frac{d^2\varphi_j}{dx^2}dx$$

$$K_{ij}^{21} = \int_{x_b}^{x_a} (EA)\left(EA + p_1\hat{N} + p_1 GAK_S\right)\frac{d\overline{w}_h}{dx}\frac{d^2\varphi_i}{dx^2}\frac{d^2\varphi_j}{dx^2}dx$$

$$K_{ij}^{22} = \int_{x_b}^{x_a} p_1\left[\left(GAK_S + \hat{N}\right)^2 \frac{d^2\varphi_i}{dx^2}\frac{d^2\varphi_j}{dx^2}dx + p_2\left(GAK_S\right)^2\frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx} + (EA)^2\left(\frac{d\overline{w}_h}{dx}\right)^2\frac{d^2\varphi_i}{dx^2}\frac{d^2\varphi_j}{dx^2}\right]dx$$

$$K_{ij}^{23} = \int_{x_b}^{x_a} p_1 GAK_S\left(GAK_S + \hat{N}\right)\left[\frac{d^2\varphi_i}{dx^2}\frac{d\varphi_j}{dx}dx + p_2 GAK_S\frac{d\varphi_j}{dx}\left(-EI\frac{d^2\varphi_j}{dx^2} + GAK_S\varphi_j\right)\right]dx$$

$$K_{ij}^{31} = \int_{x_b}^{x_a} p_1 (EA)(GAK_S)\frac{d\overline{w}_h}{dx}\frac{d^2\varphi_i}{dx^2}\frac{d^2\varphi_j}{dx^2}dx$$

$$K_{ij}^{32} = \int_{x_b}^{x_a} p_1 GAK_S\left(GAK_S + \hat{N}\right)\left[\frac{d^2\varphi_j}{dx^2}\frac{d\varphi_i}{dx}dx + p_2 GAK_S\frac{d\varphi_J}{dx}\left(-EI\frac{d^2\varphi_i}{dx^2} + GAK_S\varphi_i\right)\right]dx$$

$$K_{ij}^{33} = \int_{x_b}^{x_a} p_1\left[\left(GAK_S\right)^2\frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx} + p_2\int_{x_b}^{x_a}\left(-EI\frac{d^2\varphi_i}{dx^2} + GAK_S\varphi_i\right)\left(-EI\frac{d^2\varphi_j}{dx^2} + GAK_S\varphi_j\right)\right]dx$$

$$F_i^1 = -EA\int_{x_b}^{x_a}\left[f\frac{d^2\varphi_i}{dx^2} + qp_1\frac{d\overline{w}_h}{dx}\frac{d^2\varphi_i}{dx^2}\right]dx$$

$$F_i^2 = -\int_{x_b}^{x_a}\left[EAf\frac{d\overline{w}}{dx} + qp_1\left(GAK_S + \hat{N}\right)\right]\frac{d^2\varphi_i}{dx^2}dx$$

$$F_i^3 = -p_1 GAK_S\int_{x_b}^{x_a} q\frac{d\varphi_i}{dx}dx \tag{33}$$

From the terms of K_{ij}^{33} it is clear that the terms p_1 and p_2 should be taken such that $p_2 = p_1/h^2$, where h is the element length.

3.5 Least-Squares Finite Element of the EBT (EBT-2)

Here consider the first-order equations

$$
\begin{aligned}
-\frac{dN}{dx} - f &= 0, & \frac{N}{EA} - \left[\frac{du}{dx} + \left(\frac{dw}{dx}\right)^2\right] &= 0 \\
-\frac{dV}{dx} + \frac{d}{dx}\left(N\theta\right) - q &= 0, & \theta + \frac{dw}{dx} &= 0 \\
\frac{M}{EI} - \frac{d\theta}{dx} &= 0, & -V + \frac{dM}{dx} &= 0
\end{aligned}
\tag{34}
$$

The least-squares functional associated with the above six equations over a typical element is

$$
\begin{aligned}
J_2(u_h, w_h, \ldots) = \int_{x_a}^{x_b} \Bigg[& \left(\frac{dN_h}{dx} + f\right)^2 + \left(\frac{N_h}{EA} - \left[\frac{du_h}{dx} + \left(\frac{dw_h}{dx}\right)^2\right]\right)^2 + \left(-\frac{dV_h}{dx} + \frac{d}{dx}\left(N\theta\right) - q\right)^2 + \\
& \left(\theta_h + \frac{dw_h}{dx}\right)^2 + \left(\frac{M_h}{EI} - \frac{d\theta_h}{dx}\right)^2 + \left(-V_h + \frac{dM_h}{dx}\right)^2 \Bigg] dx
\end{aligned}
\tag{35}
$$

The minimum of J_2 is equivalent to the following 6 statements:

$$
\begin{aligned}
0 &= \int_{x_a}^{x_b} \left[-\frac{d\delta u_h}{dx} \left(\frac{N_h}{EA} - \left(\frac{du_h}{dx} + \frac{1}{2}\left(\frac{dw_h}{dx}\right)^2 \right) \right) \right] dx \\
0 &= \int_{x_a}^{x_b} \left[-\frac{d\delta w_h}{dx}\frac{dw_h}{dx} \left[\frac{N_h}{EA} - \left(\frac{du_h}{dx} + \frac{1}{2}\left(\frac{dw_h}{dx}\right)^2 \right) \right] + \frac{d\delta w_h}{dx}\left(\theta_h + \frac{dw_h}{dx}\right) \right] dx \\
0 &= \int_{x_a}^{x_b} \Bigg[\delta\theta_h \left(\theta_h + \frac{dw_h}{dx}\right) - \frac{d\delta\theta_h}{dx}\left(\frac{M_h}{EI} - \frac{d\theta_h}{dx}\right) + \left(\frac{d\delta\theta_h}{dx} N_h + \frac{dN_h}{dx}\delta\theta_h\right) * \\
&\qquad \left(-\frac{dV_h}{dx} + \frac{d\theta_h}{dx} N + \frac{dN_h}{dx}\theta_h - q\right) \Bigg] dx
\end{aligned}
\tag{36}
$$

$$
\begin{aligned}
0 &= \int_{x_a}^{x_b} \Bigg[\frac{d\delta N_h}{dx}\left(\frac{dN_h}{dx} + f\right) + \left(\frac{\delta N_h}{EA}\right)\left(\frac{N_h}{EA} - \left(\frac{du_h}{dx} + \frac{1}{2}\left(\frac{dw_h}{dx}\right)^2\right)\right) + \\
&\qquad \left(\frac{d\theta_h}{dx}\delta N + \frac{d\delta N}{dx}\delta\theta_h\right)\left(-\frac{dV}{dx} + \frac{d\theta_h}{dx} N_h + \frac{dN_h}{dx}\theta_h - q\right) \Bigg] dx \\
0 &= \int_{x_a}^{x_b} \left[\left(-\frac{\delta M_h}{EI}\right)\left(\frac{M_h}{EI} - \frac{d\theta_h}{dx}\right) + \frac{d\delta M_h}{dx}\left(-V_h + \frac{dM_h}{dx}\right) \right] dx \\
0 &= \int_{x_a}^{x_b} \left[\left(-\frac{d\delta V_h}{dx}\right)\left(-\frac{dV_h}{dx} + kw_h - q\right) - \frac{d\delta V_h}{dx}\left(-\frac{dV_h}{dx} + \frac{d\theta_h}{dx} N_h + \frac{dN_h}{dx}\theta_h - q\right) \right] dx
\end{aligned}
$$

In this model, all physical variables that enter the specification of physical boundary conditions appear as unknowns. Hence, they are all approximated by Lagrange interpolation functions:

$$\begin{aligned} u_h &= \sum_{j=1}^{m} u_j \psi_j(x), \quad w_h = \sum_{j=1}^{m} w_j \psi_j(x), \quad \theta_h = \sum_{j=1}^{m} \theta_j \psi_j(x), \\ N_h &= \sum_{j=1}^{m} N_j \psi_j(x), \quad M_h = \sum_{j=1}^{m} M_j \psi_j(x), \quad V_h = \sum_{j=1}^{m} V_j \psi_j(x) \end{aligned} \tag{37}$$

where $u_j, w_j, \theta_j, N_j, M_j$ and V_j denote the nodal values of $u_h, w_h, \theta_h, N_h, M_h$ and V_h, respectively at the jth node. Thus we obtain the following finite element model:

$$\begin{bmatrix} [K^{11}] & [K^{12}] & [K^{13}] & [K^{14}] & [K^{15}] & [K^{16}] \\ [K^{21}] & [K^{22}] & [K^{23}] & [K^{24}] & [K^{25}] & [K^{26}] \\ [K^{31}] & [K^{32}] & [K^{33}] & [K^{34}] & [K^{35}] & [K^{36}] \\ [K^{41}] & [K^{42}] & [K^{43}] & [K^{44}] & [K^{45}] & [K^{46}] \\ [K^{51}] & [K^{52}] & [K^{53}] & [K^{54}] & [K^{55}] & [K^{56}] \\ [K^{61}] & [K^{62}] & [K^{63}] & [K^{64}] & [K^{65}] & [K^{66}] \end{bmatrix} \begin{pmatrix} \{u\} \\ \{w\} \\ \{\theta\} \\ \{N\} \\ \{M\} \\ \{V\} \end{pmatrix} = \begin{pmatrix} \{F^1\} \\ \{F^2\} \\ \{F^3\} \\ \{F^4\} \\ \{F^5\} \\ \{F^6\} \end{pmatrix} \tag{38}$$

where

$$K_{ij}^{11} = \int_{x_a}^{x_b} \frac{d\psi_i}{dx}\frac{d\psi_j}{dx}dx, \qquad K_{ij}^{12} = \int_{x_a}^{x_b} \frac{1}{2}\frac{dw}{dx}\frac{d\psi_i}{dx}\frac{d\psi_j}{dx}dx$$

$$K_{ij}^{14} = -\frac{1}{EA}\int_{x_a}^{x_b} \frac{d\psi_i}{dx}\psi_j dx, \quad K_{ij}^{21} = \int_{x_a}^{x_b} \frac{dw}{dx}\frac{d\psi_i}{dx}\frac{d\psi_j}{dx}dx$$

$$K_{ij}^{22} = \int_{x_a}^{x_b} \left(\frac{d\psi_i}{dx}\frac{d\psi_j}{dx} + \frac{1}{2}\frac{d\psi_i}{dx}\frac{d\psi_j}{dx}\left(\frac{dw}{dx}\right)^2 \right) dx$$

$$K_{ij}^{23} = \int_{x_a}^{x_b} \frac{d\psi_i}{dx}\psi_j dx, \quad K_{ij}^{32} = \int_{x_a}^{x_b} \frac{d\psi_j}{dx}\psi_i dx$$

$$K_{ij}^{33} = \int_{x_a}^{x_b} \left(\psi_j\psi_i + \frac{d\psi_i}{dx}\frac{d\psi_j}{dx} + \left(\frac{d\psi_i}{dx}N + \frac{dN}{dx}\psi_i\right)\left(\frac{dN}{dx}\psi_j + N\frac{d\psi_j}{dx}\right) \right) dx$$

$$K_{ij}^{35} = -\frac{1}{EI}\int_{x_a}^{x_b} \frac{d\psi_i}{dx}\psi_j dx, \quad K_{ij}^{36} = -\int_{x_a}^{x_b} \frac{d\psi_j}{dx}\left(\frac{d\psi_i}{dx}N + \frac{dN}{dx}\psi_i\right) dx$$

$$
\begin{aligned}
&K_{ij}^{41}=-\frac{1}{EA}\int_{x_a}^{x_b}\frac{d\psi_j}{dx}\psi_i dx,\;\; K_{ij}^{42}=-\frac{1}{EA}\frac{1}{2}\int_{x_a}^{x_b}\frac{d\psi_j}{dx}\psi_i\frac{dw}{dx}dx\\
&K_{ij}^{43}=\left(\frac{d\psi_i}{dx}\bar{\theta}+\psi_i\frac{d\bar{\theta}}{dx}\right)\left(\frac{dN}{dx}\psi_j+N\frac{d\psi_j}{dx}\right)\\
&K_{ij}^{44}=\int_{x_a}^{x_b}\left(\frac{1}{(EA)^2}\psi_j\psi_i+\frac{d\psi_i}{dx}\frac{d\psi_j}{dx}\right)dx\\
&K_{ij}^{46}=-\int_{x_a}^{x_b}\frac{d\psi_j}{dx}\left(\frac{d\psi_i}{dx}\bar{\theta}+\psi_i\frac{d\bar{\theta}}{dx}\right)dx,\;\; K_{ij}^{53}=-\frac{1}{EI}\int_{x_a}^{x_b}\frac{d\psi_j}{dx}\psi_i dx\\
&K_{ij}^{55}=\int_{x_a}^{x_b}\left(\frac{1}{(EI)^2}\psi_j\psi_i+\frac{d\psi_i}{dx}\frac{d\psi_j}{dx}\right)dx,\;\; K_{ij}^{56}=-\int_{x_a}^{x_b}\frac{d\psi_i}{dx}\psi_j dx\\
&K_{ij}^{63}=-\int_{x_a}^{x_b}\frac{d\psi_i}{dx}\left(\frac{d\psi_j}{dx}N+\frac{dN}{dx}\psi_j\right)dx,\;\; K_{ij}^{65}=-\int_{x_a}^{x_b}\frac{d\psi_j}{dx}\psi_i dx\\
&K_{ij}^{66}=\int_{x_a}^{x_b}\left(\psi_j\psi_i+\frac{d\psi_i}{dx}\frac{d\psi_j}{dx}\right)dx,\;\; F_i^3=\int_{x_a}^{x_b}\left[q\left(\frac{d\psi_i}{dx}\bar{N}+\psi_i\frac{d\bar{N}}{dx}\right)\right]dx\\
&F_i^4=\int_{x_a}^{x_b}\left[-f\frac{d\psi_i}{dx}+q\left(\frac{d\psi_i}{dx}\bar{\theta}+\psi_i\frac{d\bar{\theta}}{dx}\right)\right]dx,\;\; F_i^6=-\int_{x_a}^{x_b}q\frac{d\psi_i}{dx}dx
\end{aligned}
\tag{39}
$$

3.6 Least-Squares Finite Element of TBT (TBT-2)

Here we consider the first-order equations

$$
\begin{aligned}
&-\frac{dN}{dx}-f=0, && \frac{N}{EA}-\left[\frac{du}{dx}+\left(\frac{dw}{dx}\right)^2\right]=0\\
&-\frac{dV}{dx}+\frac{d}{dx}\left(N\left(\frac{V}{GAK}-\phi\right)\right)-q=0, && \frac{V}{GAK}-\left(\frac{dw}{dx}+\phi\right)=0\\
&\frac{M}{EI}-\frac{d\phi}{dx}=0, && -V+\frac{dM}{dx}=0
\end{aligned}
\tag{40}
$$

The least-squares functional associated with the above six equations over a typical element is

$$J_2(u_h, w_h, \ldots) = \int_{x_a}^{x_b} \left[\left(\frac{dN_h}{dx} + f \right)^2 + \left(\frac{N_h}{EA} - \left[\frac{du_h}{dx} + \left(\frac{dw_h}{dx} \right)^2 \right] \right)^2 + \right.$$
$$\left[-\frac{dV}{dx} + \frac{d}{dx}\left(N\left(\frac{V}{GAK} - \phi \right) \right) - q \right]^2 + \tag{41}$$
$$\left. \left(\frac{V}{GAK} - \left(\frac{dw}{dx} + \phi \right) \right)^2 + \left(\frac{M_h}{EI} - \frac{d\theta_h}{dx} \right)^2 + \left(-V_h + \frac{dM_h}{dx} \right)^2 \right] dx$$

The necessary condition for minimum of J_2 yields the following six statements:

$$0 = \int_{x_a}^{x_b} \left[-\frac{d\delta u_h}{dx} \left(\frac{N_h}{EA} - \left(\frac{du_h}{dx} + \frac{1}{2}\left(\frac{dw_h}{dx} \right)^2 \right) \right) \right] dx$$

$$0 = \int_{x_a}^{x_b} \left[-\frac{d\delta w_h}{dx} \frac{dw_h}{dx} \left[\frac{N_h}{EA} - \left(\frac{du_h}{dx} + \frac{1}{2}\left(\frac{dw_h}{dx} \right)^2 \right) \right] + \frac{d\delta w_h}{dx} \left(\frac{V_h}{GAK} - \frac{dw_h}{dx} - \phi \right) \right] dx$$

$$0 = \int_{x_a}^{x_b} \left[\begin{array}{l} \delta\phi_h \left(\frac{V_h}{GAK} - \phi_h - \frac{dw_h}{dx} \right) - \frac{d\delta\theta_h}{dx} \left(\frac{M_h}{EI} - \frac{d\theta_h}{dx} \right) - \\ \left(\frac{d\delta\theta_h}{dx} N_h + \frac{dN_h}{dx} \delta\theta_h \right) \left(-\frac{dV_h}{dx} + \frac{dN}{dx} \frac{V_h}{GAK} + \frac{N_h}{GAK} \frac{dV_h}{dx} - N\frac{d\phi}{dx} - \phi \frac{dN}{dx} - q \right) \end{array} \right] dx$$

$$0 = \int_{x_a}^{x_b} \left[\frac{d\delta N_h}{dx} \left(\frac{dN_h}{dx} + f \right) + \left(\frac{\delta N_h}{EA} \right) \left(\frac{N_h}{EA} - \left(\frac{du_h}{dx} + \frac{1}{2}\left(\frac{dw_h}{dx} \right)^2 \right) \right) + \right.$$
$$\left(\frac{d\delta N_h}{dx} \frac{V_h}{GAK} + \frac{\delta N_h}{GAK} \frac{dV_h}{dx} - \frac{d\delta N_h}{dx} \phi - \frac{d\phi_h}{dx} \delta N_h \right) * \tag{42}$$
$$\left. \left(\frac{dN_h}{dx} \frac{V_h}{GAK} + \frac{N_h}{GAK} \frac{dV_h}{dx} - \frac{dN_h}{dx} \phi - \frac{d\phi_h}{dx} N_h - \frac{dV_h}{dx} - q \right) \right] dx$$

$$0 = \int_{x_a}^{x_b} \left[\left(\frac{\delta M_h}{EI} \right) \left(\frac{M_h}{EI} - \frac{d\theta_h}{dx} \right) + \frac{d\delta M_h}{dx} \left(-V_h + \frac{dM_h}{dx} \right) \right] dx$$

$$0 = \int_{x_a}^{x_b} \left[\frac{\delta V_h}{GAK} \left(\frac{V_h}{GAK} - \phi_h - \frac{dw_h}{dx} \right) - \delta V_h \left(-V_h + \frac{dM_h}{dx} \right) + \left(-\frac{d\delta V_h}{dx} + \frac{dN_h}{dx} \frac{\delta V_h}{GAK} + \frac{N_h}{GAK} \frac{d\delta V_h}{dx} \right) \right.$$
$$\left. * \left(-\frac{dV_h}{dx} + \frac{dN_h}{dx} \frac{V_h}{GAK} + \frac{N_h}{GAK} \frac{dV_h}{dx} - \frac{dN_h}{dx} \phi - \frac{d\phi_h}{dx} N_h \right) \right] dx$$

Once again, the model admits Lagrange type interpolation of all variables:

$$\begin{aligned} u_h &= \sum_{j=1}^{m} u_j \psi_j(x), \quad w_h = \sum_{j=1}^{m} w_j \psi_j(x), \qquad \theta_h = \sum_{j=1}^{m} \theta_j \psi_j(x), \\ N_h &= \sum_{j=1}^{m} N_j \psi_j(x), \quad M_h = \sum_{j=1}^{m} M_j \psi_j(x), \quad V_h = \sum_{j=1}^{m} V_j \psi_j(x) \end{aligned} \tag{43}$$

where $u_j, w_j, \theta_j, N_j, M_j$ and V_j denote the nodal values of $u_h, w_h, \theta_h, N_h, M_h$ and V_h, respectively at the jth node. Thus we obtain the following finite element model:

$$\begin{bmatrix} [K^{11}] & [K^{12}] & [K^{13}] & [K^{14}] & [K^{15}] & [K^{16}] \\ [K^{21}] & [K^{22}] & [K^{23}] & [K^{24}] & [K^{25}] & [K^{26}] \\ [K^{31}] & [K^{32}] & [K^{33}] & [K^{34}] & [K^{35}] & [K^{36}] \\ [K^{41}] & [K^{42}] & [K^{43}] & [K^{44}] & [K^{45}] & [K^{46}] \\ [K^{51}] & [K^{52}] & [K^{53}] & [K^{54}] & [K^{55}] & [K^{56}] \\ [K^{61}] & [K^{62}] & [K^{63}] & [K^{64}] & [K^{65}] & [K^{66}] \end{bmatrix} \begin{pmatrix} \{u\} \\ \{w\} \\ \{\phi\} \\ \{N\} \\ \{M\} \\ \{V\} \end{pmatrix} = \begin{pmatrix} \{F^1\} \\ \{F^2\} \\ \{F^3\} \\ \{F^4\} \\ \{F^5\} \\ \{F^6\} \end{pmatrix} \tag{44}$$

where

$$K_{ij}^{11} = \int_{x_a}^{x_b} \frac{d\psi_i}{dx}\frac{d\psi_j}{dx}dx$$

$$K_{ij}^{12} = \int_{x_a}^{x_b} -\frac{1}{2}\frac{dw}{dx}\frac{d\psi_i}{dx}\frac{d\psi_j}{dx}dx$$

$$K_{ij}^{14} = -\frac{1}{EA}\int_{x_a}^{x_b} \frac{d\psi_i}{dx}\psi_j dx$$

$$K_{ij}^{21} = \int_{x_a}^{x_b} \frac{dw}{dx}\frac{d\psi_i}{dx}\frac{d\psi_j}{dx}dx$$

$$K_{ij}^{22} = \int_{x_a}^{x_b} \left(\frac{d\psi_i}{dx}\frac{d\psi_j}{dx} + \frac{1}{2}\frac{d\psi_i}{dx}\frac{d\psi_j}{dx}\left(\frac{dw}{dx}\right)^2 \right) dx$$

$$K_{ij}^{23} = \int_{x_a}^{x_b} \frac{d\psi_i}{dx}\psi_j dx$$

$$K_{ij}^{24} = -\int_{x_a}^{x_b} \frac{dw}{dx}\frac{1}{EA}\frac{d\psi_i}{dx}\psi_j dx$$

$$K_{ij}^{26} = -\int_{x_a}^{x_b} \frac{1}{GAK}\frac{d\psi_i}{dx}\psi_j dx$$

$$K_{ij}^{32} = \int_{x_a}^{x_b} \frac{d\psi_j}{dx}\psi_i dx$$

$$K_{ij}^{33} = \int_{x_a}^{x_b} \left(\psi_j\psi_i + \frac{d\psi_i}{dx}\frac{d\psi_j}{dx} + \left(\frac{d\psi_i}{dx}N + \frac{dN}{dx}\psi_i \right)\left(\frac{dN}{dx}\psi_j + N\frac{d\psi_j}{dx} \right) \right) dx$$

$$K_{ij}^{35} = -\frac{1}{EI}\int_{x_a}^{x_b} \frac{d\psi_i}{dx}\psi_j dx$$

$$K_{ij}^{36} = \int_{x_a}^{x_b} \left(-\frac{d\psi_j}{dx} + \frac{\psi_j}{GAK}\frac{dN}{dx} + \frac{N}{GAK}\frac{d\psi_j}{dx} \right)\left(\frac{d\psi_i}{dx}N + \frac{dN}{dx}\psi_i \right) + \frac{\psi_j}{GAK}\psi_i dx$$

$$K_{ij}^{41} = -\frac{1}{EA}\int_{x_a}^{x_b} \frac{d\psi_j}{dx}\psi_i dx$$

$$K_{ij}^{42} = -\frac{1}{EA}\int_{x_a}^{x_b} \frac{d\psi_j}{dx}\psi_i \frac{1}{2}\frac{dw}{dx}dx$$

$$K_{ij}^{44} = \int_{x_a}^{x_b} \left(\begin{array}{l} \frac{1}{(EA)^2}\psi_j\psi_i + \frac{d\psi_i}{dx}\frac{d\psi_j}{dx} + \left(\frac{\bar{V}}{GAK}\frac{d\psi_i}{dx} + \frac{\psi_i}{GAK}\frac{d\bar{V}}{dx} - \frac{d\psi_i}{dx}\phi - \frac{d\phi}{dx}\psi_i \right) \\ \left(\frac{\bar{V}}{GAK}\frac{d\psi_j}{dx} + \frac{\psi_j}{GAK}\frac{d\bar{V}}{dx} - \frac{d\psi_j}{dx}\phi - \frac{d\phi}{dx}\psi_j \right) \end{array} \right) dx$$

$$K_{ij}^{46} = -\int_{x_a}^{x_b} \frac{d\psi_j}{dx}\left(\frac{\bar{V}}{GAK}\frac{d\psi_i}{dx} + \frac{\psi_i}{GAK}\frac{d\bar{V}}{dx} - \frac{d\psi_i}{dx}\phi - \frac{d\phi}{dx}\psi_i \right) dx$$

$$K_{ij}^{53} = -\frac{1}{EI}\int_{x_a}^{x_b} \frac{d\psi_j}{dx}\psi_i dx$$

$$K_{ij}^{55} = \int_{x_a}^{x_b} \left(\frac{1}{(EI)^2}\psi_j\psi_i + \frac{d\psi_i}{dx}\frac{d\psi_j}{dx} \right) dx, \quad K_{ij}^{56} = -\int_{x_a}^{x_b} \frac{d\psi_i}{dx}\psi_j dx$$

$$K_{ij}^{62} = -\int_{x_a}^{x_b} \frac{d\psi_j}{dx}\frac{\psi_i}{GAK}$$

$$K_{ij}^{63} = \int_{x_a}^{x_b} -\frac{\psi_i\psi_j}{GAK} + \left(-\frac{d\psi_i}{dx} + \frac{dN}{dx}\frac{d\psi_i}{dx} + \frac{N}{GAK}\frac{d\psi_i}{dx}\right)\left(\psi_j\frac{dN}{dx} + N\frac{d\psi_j}{dx}\right)dx$$

$$K_{ij}^{65} = -\int_{x_a}^{x_b} \frac{d\psi_j}{dx}\psi_i dx$$

$$K_{ij}^{66} = \int_{x_a}^{x_b} \left[\left(\frac{1}{(GAK)^2}\psi_j\psi_i + \psi_j\psi_i\right) + \left(-\frac{d\psi_i}{dx} + \frac{dN}{dx}\frac{d\psi_i}{dx} + \frac{N}{GAK}\frac{d\psi_i}{dx}\right)\right.$$
$$\left.\left(-\frac{d\psi_j}{dx} + \frac{dN}{dx}\frac{\psi_j}{GAK} + \frac{N}{GAK}\frac{d\psi_j}{dx}\right)\right]dx$$

$$F_i^4 = \int_{x_a}^{x_b} \left[-f\frac{d\psi_i}{dx} + q\left(\frac{\bar{V}}{GAK}\frac{d\psi_i}{dx} + \frac{\psi_i}{GAK}\frac{d\bar{V}}{dx} - \frac{d\psi_i}{dx}\phi - \frac{d\phi}{dx}\psi_i\right)\right]dx$$

$$F_i^3 = \int_{x_a}^{x_b} \left[-q\left(\frac{d\psi_i}{dx}N + \frac{dN}{dx}\psi_i\right)\right]dx$$

$$F_i^6 = -\int_{x_a}^{x_b} q\frac{d\psi_i}{dx}dx \tag{45}$$

This completes the development of least squares finite element models of Euler-Bernoulli and Timoshenko beam theories.

The nonlinear finite element equations in models EBE-1, EBE-2, TBT-1, and TBT-2 are solved using the Newton-Raphson iterative method, in which the incremental solution vector $\{\delta U\}$ is computed from

$$\left\{T\left(\{U\}^{(r)}\right)\right\}.\{\delta U\} = -\left\{R\left(\{U\}^{(r)}\right)\right\} \tag{46}$$

where the tangent stiffness coefficients $T_{ij}^{\alpha\beta}$ are computed using the definition,

$T_{ij}^{\alpha\beta} = \left(\frac{\partial R_i^\alpha}{\partial \Delta_j^\beta}\right)^{(r)}$ and the superscript r refers to the previous iteration number, and the total solution at (r+1)st iteration is given by $\{U\}^{(r+1)} = \{U\}^{(r)} + \{\delta U\}$.

4. NUMERICAL RESULTS

The following example is considered for EBT and TBT model for conventional weak form and least-squares models. We consider a beam of length L, subjected to a uniformly distributed load q = 10 lb/in which is applied in 10 load steps in the nonlinear analysis. The following data is used in obtaining numerical results:

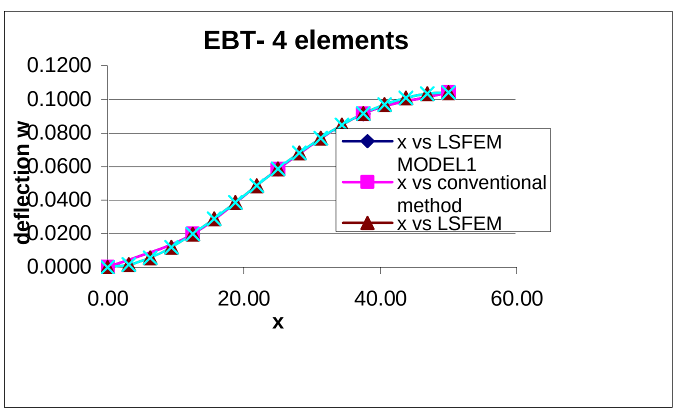
$E = 30 \times 10^6$ psi, $L = 100$ in, $A = 1\times1$ in^2, tolerance = 0.001, max. iterations = 30

Boundary conditions used are (1) both ends hinged, (2) both ends clamped, and (3) both ends pinned (see Reddy [2]). Only selective results are discussed here.

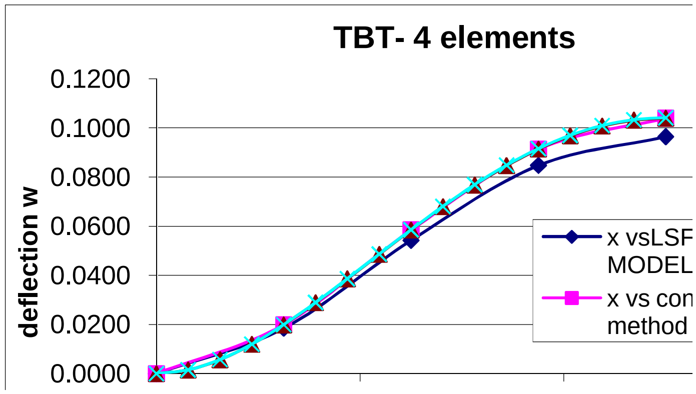
Table 1 contains a comparison of linear solutions obtained by various models using 4 elements in the half beam. Plot of x vs. deflection (w) of a beam clamped at both ends for different models are shown in Figures.1 and 2, using a mesh of 4 and 32 elements, respectively (results of the 8 element mesh are now shown here).

Table 1. Comparison of displacements and forces for hinged-hinged beam

	EBT		TBT		
Model	***x***	***w***	***dw/dx***	***w***	$\boldsymbol{\phi}$
Conven.	50.00	0.521	0.000	0.508	0.000
	x	*du/dx*	*w*	*M*	*V=dM/dx*
EBT-1	50.00	0.0000	0.4756	0.0000	~0.0005
	x	*w*	θ or ϕ	*M*	*V*
EBT-2	50.00	0.5208	0.0000	1250.00	0.0000
TBT-2	50.00	0.5208	0.0000	1250.00	0.0000

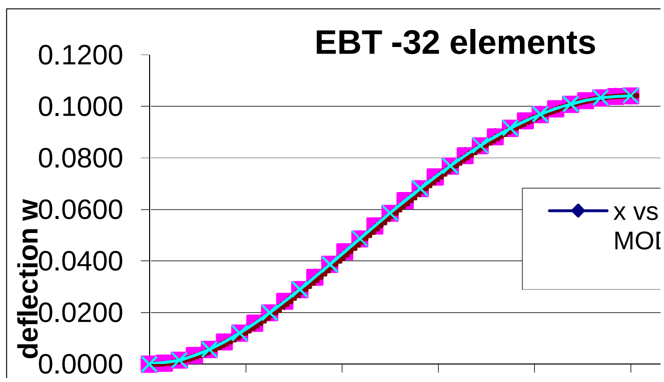


(a). x vs. deflection for different models for EBT

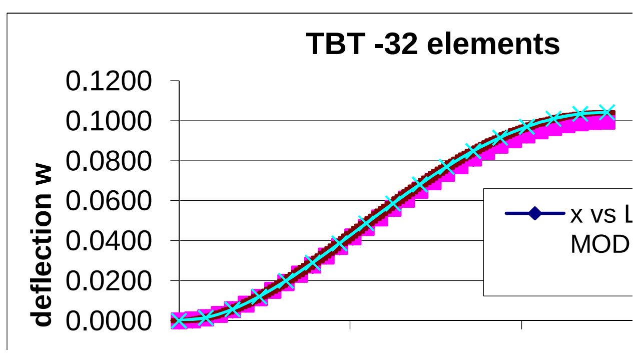


(b). x vs. deflection for different models for TBT

Figure 1. Plots of x vs. deflection in different models of EBT and TBT for a clamped beam.



(a). x vs. deflection for different models for EBT



(b). x vs. deflection for different models for TBT

Figure 2. Plots of x vs. deflection in different models of EBT and TBT for a clamped beam (32 elements)

A comparison of q vs. maximum deflection obtained with the EBT and TBT models for the nonlinear beams are shown in Figures 3 and 4 for clamped and pinned beams, respectively. A comparison of x vs. shear force for LSFEM MODEL2 and conventional model are shown in Figure 5 for the pinned beam. The shear forces obtained with LSFEM MODEL2 follow a smooth curve whereas they are discontinuous at the internal nodes for the conventional model.

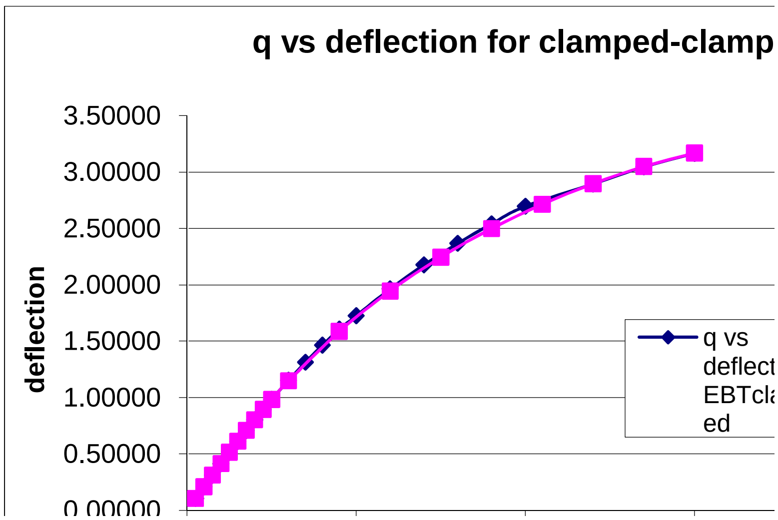


Figure 3. Comparison of q vs. maximum transverse deflection for EBT and TBT for clamped-clamped boundary conditions

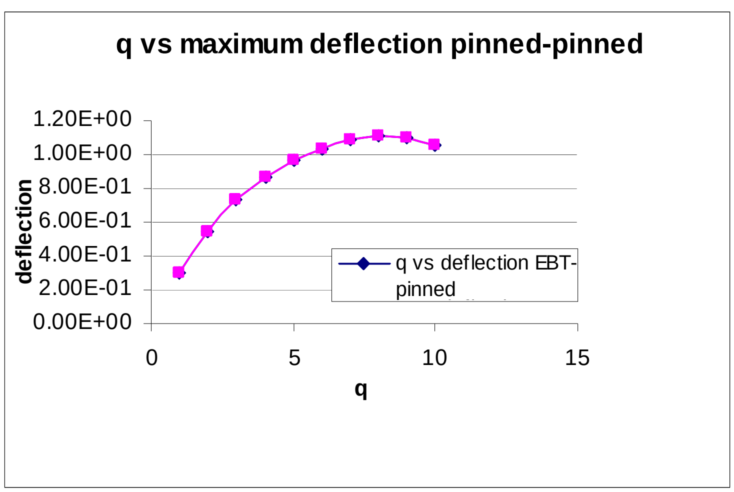


Figure 4. Comparison of q vs. maximum transverse deflection for EBT and TBT for pinned-pinned boundary conditions

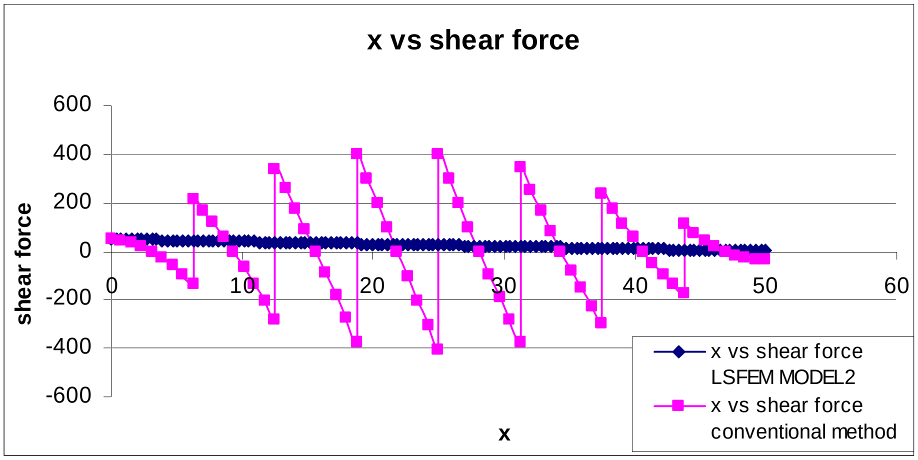


Figure 5. Comparison of x vs. shear force for LSFEM MODEL2 and conventional model

5. CONCLUSIONS

From the results presented in Section 4, the following observations and conclusions can be made.

1) The plots of x vs. deflection for EBE-1, TBT-1, and conventional models closely fit the exact solution curve. A good solution accuracy for deflection of TBT-1 can be observed even for small number of elements for various boundary conditions. As the number of elements is increased, the solutions match closely with the exact solutions for different boundary conditions.
2) The least-squares models contain forces and moments as primary variables and therefore yield increased accuracy for the variables when compared to the conventional displacement finite element models.
3) Another salient feature of least-squares models is that they always lead to a positive-definite system of equations, which allow the use of fast iterative methods for solution.

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