PERFORMANCE CONTROL FOR EFFICIENT DESIGN OF REGULAR GRILLAGES UNDER UNIFORM LOADING

M. Grigorian[*]
MGA, Structural Engineering Inc. 111 N. Jackson St. Glendale, CA.91206, USA

ABSTRACT

Performance control (PC) is a new design philosophy that aims at rational and efficient selection of structural members rather than probing their usefulness through iterative proceses. The basic notion behind PC is that structural response is mainly a function of design and detailing, rather than numerical analysis. PC is a design approach in which the properties of the structural elements are selected in accordance with predetermined performance related objectives, such as limiting displacements at first yield and/or at incipient collapse, rather than compared against arbitrary criteria. PC procedures result in highly predictable structural behavior and economically efficient designs for the class of regular, space frames considered in this paper. Neither irregular boundary conditions nor non-uniform loading have been addressed in this paper. The proposed methodology is suitable for both manual as well as spreadsheet computations. The applications of the proposed solutions have been illustrated through a number of generic examples.

Keywords: Space structures; regular grids; performance control; plastic design; displacements at failure; load sharing

1. INTRODUCTION

Multi-member space frames are ideally suited for plastic limit state design. The plastic design of grillages [1-8], interconnected trusses [9-12] and similar systems [13, 14] has been the subject of ongoing studies since early1950s. Notwithstanding, the progress and widespread use of the concept has been hindered due to absence of practical methods of plastic displacement analysis, for decades. The manual computation of plastic deformations of indeterminate structures, especially at the onset of collapse, remains one of the most challenging [14-18] aspects of space frame design. However, recent advances [19-21] in the

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[*] E-mail address of the corresponding author: Markarjan@aol.com (M. Grigorian)

plastic design of moment frames and similar structures have helped revive interest in the manual as well as computer analysis of plastic deformations at incipient failure. The ability to predict plastic response at first yield and at incipient collapse empowers the engineer to design otherwise complicated structural systems with reference to several, predictable target displacements. Most importantly, it provides a simple means of plotting accurate, idealized load-displacement (pushdown) curves for grillage type space structures considered in this paper.

In the forthcoming presentation attention is focused on design analysis, rather than analyzing designs. Traditional plastic analysis and design methods involve the investigation of best-guessed members for specific loading with arbitrary displacement limits, and may entail several iterations before a desirable solution is established. Plastic design analysis, on the other hand, avoids generalities and focuses attention on structure/loading specific response modes at target conditions. By performing design analysis, as opposed to analysing designs, most unnecessary guesswork and substantial amounts of preliminary computations can be avoided. In other words the design effort may be directed towards rational selection rather than routine investigation. Design analysis is the basis of the recently introduced performance control [22-24, 36] for the design of regular moment frames under seismic loading.

PC is somewhat reminiscent of the pushover analysis that is commonly used to predict the response of upright structures under extreme lateral loading conditions. PC, on the other hand, utilizes the same concepts to design ductile structural systems under similar conditions with a view towards pre-assigned target displacements, and, as such may also be classified as a displacement based [25] pushdown procedure or a performance based plastic design [26] for horizontally arranged three dimensional systems. Two specific target displacements and their limiting values are introduced as part of development of PC. Maximum transverse displacements at first yield are obtained through well established elastic finite difference analysis[27, 28]. The corresponding displacements at the onset of collapse are then related to the positions of formation of the first and last sets of plastic hinges within the framework.

The development of PC for any structural configuration, whether 2D or 3D, depends on the ability to predict accurately the maximum displacements at incipient collapse. Therefore, the paper primarily focuses attention on the establishment of an algorithm for the determination of maximum, transverse, nodal plastic deformations as part of development of PC procedures for ductile gridworks under monotonously increasing normal joint forces. It has been shown that the entire loading history of the subject systems, i.e., the idealized elastic-plastic force-displacement relationship starting from zero up to first yield, from first yield up to incipient collapse and the upper ductility limit can be constructed using only two points of reference, normal nodal displacements at first yield and at incipient collapse. In PC, structural response is controlled rather than investigated. To appreciate the essence of this contribution, suffice to consider the possibilities arising from the use of the yield line analysis [29, 30] in association with displacement data at incipient collapse. Yield line theory augmented with corresponding displacement analysis could still result in more rational as well as economical flat plate/slab designs compared to those using the currently popular strip and/or equivalent frame methods of analysis. Needless to remark that the current discussion is closely related to yield line philosophies introduced some 80 years ago.

Finally, as in many regular grillage studies, the use of the finite difference calculus has considerably facilitated the formulation and solution of the constitutive equations revisited in this paper.

1.1 Basic assumptions
The following basic assumptions and design conditions are fundamental to the development and implementation of PC methodologies, that;
- the structure will not fail prematurely due to formation of inactive plastic hinges,
- neither the conditions of static equilibrium, boundary support restrains nor the yield criteria are violated within the members of the structure,
- neither local nor global stability of the sets of the intersecting beams is compromised,
- the prescribed target displacements are not exceeded,
- the plastic hinge rotations can take places without any restricting effects from the connections or other components, and
- the differences between nodal and corresponding inter-nodal displacements can be ignored for practical design purposes. See Appendix B.

It has also been assumed, for the sake of expediency, that the effects of plastic hinge offsets from beam center lines, shear strain, yield over strength and strain hardening can conservatively be ignored for practical design purposes.

2. THORETICAL DEVELOPMENT

The purpose of this section is not only to propose a new closed form solution for a frequently occurring design problem, i.e., simply supported twistless grids under uniform loading, but also to introduce the applications of PC to grillage type structures in general.

The theoretical procedures expounded in this section are presented in three distinct but related parts. In part 1, an effort has been made through classical methods of analysis, to establish an exact relationship between the maximum normal nodal displacement at first yield, Y_Y and the corresponding distributed force P_Y. The essence of this contribution is presented in Parts 2 and 3. Part 2 discusses a method of determination of the collapse load P_C. Part 3 introduce an algorithm for the computation of the plastic component of the same displacement Y_C at incipient collapse. Suffices Y and C refer to first yield and incipient collapse respectively.

This implies that the first plastic hinge to form at or near the centre of the grillage would most likely correspond to the larger of the two bending moment ratios M_Y / M^P and N_Y / N^P at $x = ma/2$ and $y = nb/2$, where

$$M_Y = \frac{4P_Y a}{2mn} \sum_{i=1,3}^{m-1} \sum_{j=1,3}^{n-1} \frac{1}{L_{ij}} \left[\frac{\sin^2 \alpha}{2 + \cos \alpha} + \cos \alpha - 1 \right] \sin \frac{i\pi}{2} \cdot \sin \frac{j\pi}{2} \tag{1a}$$

$$N_Y = \frac{4P_Y b}{2mn} \sum_{i=1,3}^{m-1} \sum_{j=1,3}^{n-1} \frac{1}{L_{ij}} \left[\frac{\sin^2 \beta}{2 + \cos \alpha\beta} + \cos\beta - 1 \right] \sin\frac{i\pi}{2} \cdot \sin\frac{j\pi}{2} \tag{1b}$$

M^P and N^P stand for plastic moments of resistance of the x and y direction beams, respectively. The coefficient of orthotropy is defined as $\mu = N^P / M^P$.
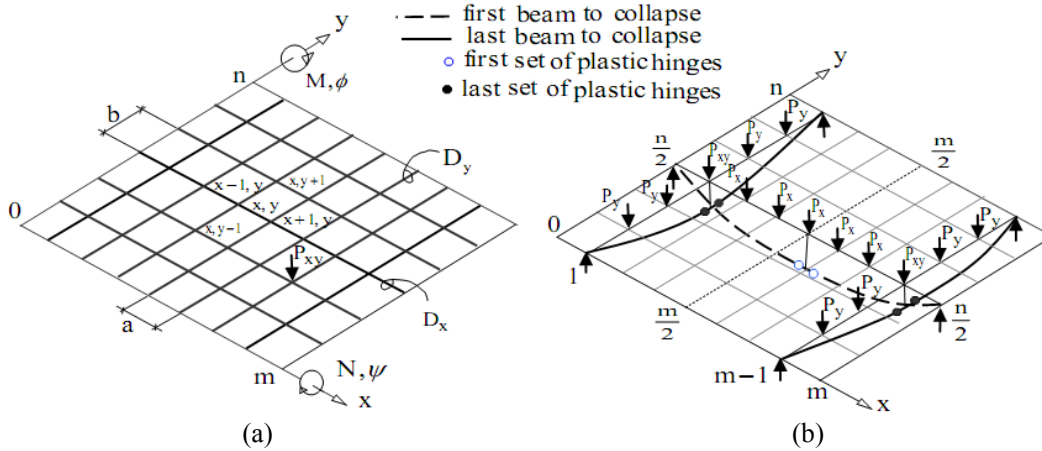

Figure 1. (a) Regular rectangular grid, co-ordinates layout and loading; (b) Maximum displacements, properties and incipient collapse

2.2 Illustrative design example 1
Consider the controlled design of a simply supported, isotropic, regular, twistless grillage under uniform distribution of normal nodal forces P_Y such that its maximum elastic displacement does not exceed a certain allowable ratio $L_{\max} / \xi_Y$. $L_{\max} = ma$ or nb whichever is the longer side.

ξ_Y is a prescribed limit at first yield. For this particular case, consider the maximum central displacement of a square, $m = n = 4$ grid at first yield. Assume the load factor is unity, $\lambda = 1$, and $\rho = \mu = 1$. Let $D_x = D_y = D = EI / a^3$, $a = b$ and $M^P = N^P$.

Solution: Eqs. (1a) and (1b) give; $Y_Y = 1.8974 P_Y a^3 / EI$ and $M_Y = M^P = 1.1716 P_Y a$ respectively. Now if $P = P_Y$ and $Y_Y \le 4a / \xi_Y$, then $I \ge 0.4048 \xi_Y a M^P / E$, would present a safe and economical design for the subject grillage. The solution can be said to be related to the formation of the first set of plastic hinges developed simultaneously on both sides of the central joint (positions 1 and 4, Figure. 2a. While the exact elastic solution of the subject grillage depends on the satisfaction of the corresponding constitutive equation, Eq. (A1) of Appendix A, and the relevant boundary support conditions, the determination of its collapse load requires the establishment of a plausible failure mode together with the satisfaction of the yield criteria $M_x \le M^P$ and $N_y \le N^P$, as well as the equilibrium and kinematic

boundary conditions. Once an acceptable plastic failure mechanism, satisfying the requirements of the uniqueness theorem [31, 32] has been established, an attempt can be made to determine a physical relationship between *the first and the last sets* of plastic hinges forming within the members of the grillage. Two such modes of collapse, pertaining to odd and even numbers of beams are depicted in Figures. 2a and 2b respectively. The numerals 1, 2 etc. refer to a plausible sequence of formation of the active plastic hinges.

2.3 Part 2- Plastic collapse load analysis
It is highly probable, in multimember space frames, to encounter situations of over-collapse with many rotationally inactive plastic hinges that neither participate nor contribute toward the development of the failure mechanism. However, for the purposes of ultimate load studies it is sufficient to engage only the active plastic hinges that are needed to generate kinematically admissible collapse patterns, without due regards to the sequence of formation of active/inactive sets of plastic hinges. Figures. 2a and 2b depict two such plausible failure patterns for the purposes of the current presentation. Let the doubly symmetric patterns of Figures. 2a and 2b, with maximum central displacements Z_P depict the true failure mechanisms of the grillage at collapse load $P = P_C$. It may therefore, be argued that each quadrant of the grid can sink into a virtual curved space defined by

$$Z_{Pxy} = \frac{4xyZ_P}{(m^2 + \delta_1^m)(n^2 + \delta_1^n)ab} \tag{2a}$$

$$\theta_{xy} = \nabla_x Z_{Pxy} = \frac{4xyZ_P}{(m^2 + \delta_1^m)(n^2 + \delta_1^n)ab} \quad \text{and} \quad \gamma_{xy} = \nabla_y Z_{Pxy} = \frac{4xyZ_P}{(m^2 + \delta_1^m)(n^2 + \delta_1^n)ab} \tag{2b}$$

The finite difference operators Δ_x and ∇_x are defined in Appendix A. θ_{xy} and γ_{xy} are the corresponding virtual rotations about *y* and *x-axis* respectively. $\delta_1^m = [(-1)^m - 1]/2$, i.e., $\delta_1^m = -1$ for *m*=odd and $\delta_1^m = 0$ for *m*=even. This allows the virtual work equation for the collapsing grid to be expressed as:

$$4\sum_{x=1}^{m/2}\sum_{y=1}^{n/2} P_C Z_{xy} = 4\sum_{y=1}^{n/2} M^P \theta_{xy} + 4\sum_{x=1}^{m/2} N^P \gamma_{xy} \tag{2c}$$

which yields

$$P_C = \left[\frac{1}{(m^2 + \delta_1^m)a} + \frac{\mu}{(n^2 + \delta_1^n)b}\right]8M^P \tag{2d}$$

as an indication of the plastic limit state load carrying capacity of the structure. Now if P_{Cx}

and P_{Cy}, are the components of P_C at collapse, then

$$P_{Cx} = \frac{P_C}{\left[1 + \frac{a\mu(m^2 + \delta_1^m)}{b(n^2 + \delta_1^n)}\right]} \qquad \text{and} \qquad P_{Cy} = \frac{P_C}{\left[1 + \frac{b(n^2 + \delta_1^n)}{a\mu(m^2 + \delta_1^m)}\right]} \qquad (2e)$$

Eq. (2d) presents a unique solution that may be verified through the use of the load sharing theorem [33, 34].
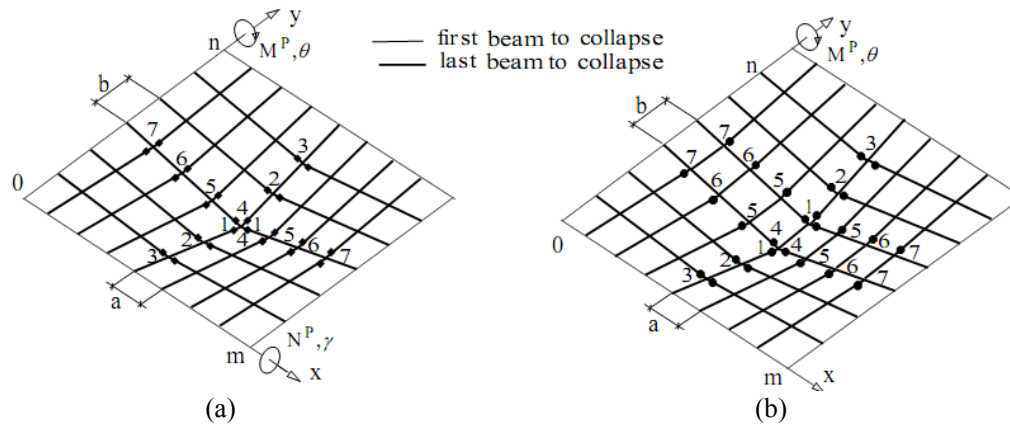

Figure 2. (a) Collapse mechanism, n being even; (b) Collapse mechanism, n being odd

The determination of maximum displacements associated with P_C is a function of the location of the first and last sets of plastic hinges forming within the members of the grid and is discussed in the following section.

2.4 Part 3- Maximum displacements at incipient collapse
However, since from a practical design point of view, displacements at first yield and incipient collapse are of greater importance than their interim counterparts, a simpler method can be devised to study the elastic-plastic performance of regular grid systems under uniform loading. This may be achieved by computing the maximum displacements due to the development of the first and last sets of plastic hinges formed within the structure. Figure.1b shows the locations of the first and last set of plastic hinges together with the deformed shape of the last stable beams, without reference to the sequence of formation of intermediate sets of hinges.
The key to the successful computation of plastic displacements at incipient collapse is in the exact determination of the bending moment distributions and the location of the last set of plastic hinges prior to failure. Both of these issues may be addressed through the rational application of the elastic state in Eqs. (1a) and (1b) to the stable condition of the structure prior to complete failure. Eq. (1a) can be uncoupled to describe the compatible displacements of the last stable beam, containing the anticipated positions of the last sets of

hinges, and the failed intersecting beam at right angles containing the set of plastic hinges formed at first yield, i.e.;

$$D_x(\Delta_x - \nabla_x)Y_{C,x} = \overline{M}_{xy} \qquad \text{and } D_y(\Delta_y - \nabla_y)Y_{C,y} = \overline{N}_{\overline{x}y} \tag{3a}$$

Consequently;

$$D_x(\Delta_x^{-1} \times \nabla_x^{-1})\overline{M}_{xy} = Y_{Cx} \qquad \text{and } D_y(\Delta_y^{-1} \times \nabla_y^{-1})\overline{N}_{xy} = Y_{Cy} \tag{3b}$$

$$(\Delta_x - \nabla_x)\overline{M}_{xy} = -P_{Cx} \qquad \text{and } (\Delta_y - \nabla_y)\overline{N}_{xy} = -P_{Cy} \tag{3c}$$

Eqs. (3a) lead to the following simple bending moment distributions at the onset of collapse;

$$\overline{M}_{xy} = \frac{4M^P(mx - x^2)}{(m^2 + \delta_1^m)} \qquad \text{and } \overline{N}_{xy} = \frac{4N^P(ny - y^2)}{(n^2 + \delta_1^n)} \tag{3d}$$

$\overline{M}_{xy}$ and $\overline{N}_{xy}$ correspond to M_{xy} and N_{xy} respectively at plastic limit state. Eqs. (3d) can also be used to determine the distributions of reactions along the supported edges [35].

$$R_{x=0} = R_{x=m} = \nabla_x \frac{M_x}{a}\Big|_{x=a} = \frac{4M^P(m-1)}{(m^2 + \delta_1^m)a} \text{ and}$$

$$R_{y=0} = R_{y=n} = \nabla_y \frac{N_x}{b}\Big|_{y=b} = \frac{4n^P(n-1)}{(n^2 + \delta_1^n)b} \tag{3e}$$

Substitution of (3d) into (3c) results in the confirmation of Eq. (2d). Substitution of Eq. (3d) in Eq. (3a) gives, after performing the pertinent difference operations we obtain

$$Y_{Cx} = \frac{P_{Cx}}{6D_x}\left\{\left[\left(\frac{x}{m}\right) - \left(\frac{x}{m}\right)^3\right]\sum_{i=1}^{m-1}(m-i) + \sum_{i=1}^{m-1}(x-i)^3 - \left(\frac{x}{m}\right)\sum_{i=1}^{m-1}(m-i)^3\right\} \tag{3f}$$

$$Y_{Cy} = \frac{P_{Cy}}{6D_y}\left\{\left[\left(\frac{y}{n}\right) - \left(\frac{y}{n}\right)^3\right]\sum_{j=1}^{n-1}(n-j) + \sum_{j=1}^{n-1}(y-j)^3 - \left(\frac{y}{n}\right)\sum_{j=1}^{n-1}(n-j)^3\right\} \tag{3g}$$

as the uncoupled displacement equations of the x and y direction beams provided that $x>i$ and $y>j$. Displacement Y_{Cy} is a maximum at x=a, y=bn/2 and x=(m−1)a, y=bn/2, i.e.

$$Y_{Cy} = \frac{P_{Cy}f_x}{384D_y}(n^2 + \delta_1^n)\left[5(n^2 - \delta_1^n) - 4\right]$$
(3h)

According to the load sharing concept (see [33, 34]), which is an interpretation of Eqs. (2d) and (2e):

"The collapse load intensity of regular twistless grillages is the sum of the collapse load intensities of two typical intersecting beams."

The load sharing concept suggests, inherently, that if the global failure load of the grillage can be computed as the sum of two physically well defined components then the magnitude of the corresponding displacements should also be related to similarly meaningful components. In other word, $P_C = P_{Cx} + P_{Cy}$, implies that the maximum transverse displacement of the grillage at incipient collapse may be computed by adding the maximum transverse displacements of two such intersecting beams, i.e.,

$$Y_C = f_y \times Y_{Cy} + f_x \times Y_{Cx}$$
(3i)

where, f_x and f_y are as yet unknown dimensionless multipliers. However, the best way to determine f_x and f_y, with fewer theoretical complications, is to link the displacements of the first and last yielding beams by the use of the virtual work equation;

$$1 \times Y_{Cx} = \frac{1}{6D_x a}\sum_{x=1}^{\bar{x}}\left[M_x(3 - \nabla_x) + (1 - \nabla_x)M_x(3 - 2\nabla_x)\right] \times m_x^* \equiv$$
$$\sum m_x^*\theta + \int(M_x/EI)m_x^*ds$$
(3k)

where m_x^* is any distribution of bending moments in equilibrium with the unit load $f_x = 1$ applied in the same sense and location, say $x=m/2$, $y=n/2$, for both m and n as even numbers, where Y_{Cx} is to be computed. For m and/or n as odd integers $f_x = 1$ would be applied at centrally/doubly symmetric node $x= (m-1)/2$, $y= (n-1)/2$. Since at incipient collapse, the last set of plastic hinges associated with rotation θ are just forming, they may be set to zero in Eq. (3k) to estimate the desired displacement as;

$$Y_{Cx} = \frac{-M^P}{48mD_x a}(m - 2)(3m^2 + 6m - 8)f_x \text{ for } m \text{ being even}$$
(3m)

$$Y_{Cx} = \frac{N^P b^2}{144(m+1)EI}(9m^3 + m^2 - 185m + 303)f_x \text{ for } n \text{ being odd}$$
(3n)

The effect of the virtual force $f_x = 1$ at the central node of the first yielding member is reflected as $f_y = m/2$ and $f_y = (m-1)/2$ at ($x=a$, $y=bn/2$) for m being even and m being odd respectively. The multiplier $f_y = (m + \delta_1^m)/2$ in Eq. (3p) reflects the effect of the unit load at $x=am/2$, $y=bn/2$. A clarification of this condition, for m=even, is presented in Appendix C. Having established exact solutions for Y_{Py} and Y_{Px} it can be shown that;

$$Y_C = \left(\frac{m + \delta_1^m}{2}\right)Y_{Cy} + Y_{Cx} = \left(\frac{m + \delta_1^m}{2} - 1\right)Y_{Cy} + (Y_{Cy} + Y_{Cx}) \qquad (3p)$$

as the exact, closed form, maximum normal nodal displacement prior to collapse . Y_{Cy} is the maximum nodal transverse displacement of a prismatic simply supported beam under n or (n-1) equidistantly applied normal nodal forces P_{Cy}, Y_{Cx} is the tip displacement of the x-direction beam under similarly distributed normal nodal forces P_{Cx}, and suspended from the supporting y-direction beam as shown in Figures (2a) and (3). $mY_{Cy}/2$ or $(m-1)Y_{Cy}/2$, depending on m being even or odd, is the rigid body tip displacement of the same beam pivoting about the point of maximum displacement of the supporting cross beam. Hence, it may be stated that:

The maximum normal nodal displacements of simply supported, regular, rectangular, twistless grids under uniformly distributed normal joint forces at incipient collapse, is given by the sum of the maximum central displacements of the first and last yielding beams plus the rigid body displacement of the former beam pivoting about the common joint of the two beams.

A graphical interpretation of this statement as well as Eq. (3p) is presented in Figure.3.The preceding statement pertaining to maximum displacements at incipient collapse may be restated as:
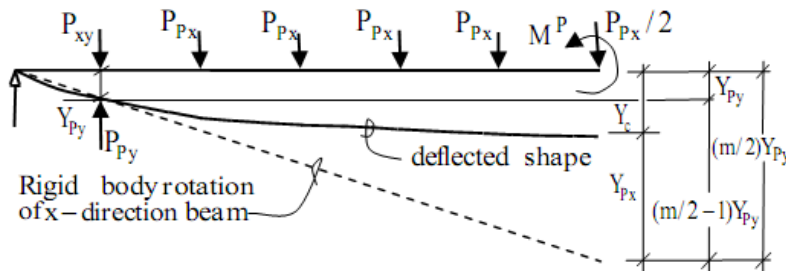
Figure 3. Components of transverse displacement of beam containing the first set of plastic hinges (m bieng even)

The maximum normal nodal displacements of simply supported, regular, rectangular, twistless grids under uniformly distributed normal joint forces, at incipient collapse, is given

by the $(m + \delta_1^m)/2$ times the maximum central displacement of the last yielding beams minus the maximum displacement of the first yielding beam.

For the sake of brevity only grids with m and n even have been addressed in the remainder of this paper.

2.5 Illustrative design example 2

Consider the controlled design of the grillage of the previous example, such that its maximum nodal displacement at incipient collapse does not exceed a certain allowable ratio $L_{\max}/\xi_C$, where ξ_C is a prescribed limit at incipient collapse.

Solution: For m=4, i.e., m even $\delta_1^4 = 0$. Eqs. (2d) and (2e) give; $M^P = P_C a m^2 / 16 = P_C a$ and $P_{Cx} = P_{Cy} = P_C / 2$ respectively. Eqs. (3h) and (3m) yield upon substitution; $Y_{Cy} = 19 P_C a^3 / 6EI$ and $Y_{Cx} = -4M^P a^2 / 6EI = -2 P_C a^3 / 3EI$ respectively, i.e.

$Y_C = (19 P_{Cy} a^3 / 6EI) \times (m/2) - 2 P_C a^3 / 3EI = 2.5 P_C a^3 / EI$, compared with $Y_Y = 1.8974 P_Y a^3 / EI$ at first yield. Therefore, $I \geq 5\xi_C a M^P / 8E$. Obviously the final design would depend on the selection of the larger value associated with ξ_Y or ξ_C. Both solutions offer safe designs since the grillage is still capable of undergoing larger displacements before failure.

2.6 Illustrative design example 3

The purpose of example 3 is to illustrate that the seemingly complex problem of elasto-plastic grillage displacements can be reduced to the insertion of simple digits in the appropriate formulae presented in this paper.

Compute the maximum elastic and plastic transverse displacements of an $m = 5 \times n = 6$, equal mesh, isotropic grillage.

Solution: $\delta_1^{m=5} = -1$ and $\delta_1^{n=6} = 0$. Eq. (1b) gives for i=1,3 and j=1,3,5; $Y_Y = 3.2278 P_Y a^3 / EI$. Eq. (2d) gives; $M^P = 9 P_C a / 5$. From Eq. (2e) we have $P_{Cx} = 3M^P / 9a = 3P_C / 5$ and $P_{Cy} = 2M^P / 9a = 2P_C / 5$. Since n>m then the first set of plastic hinges will form simultaneously at x=(m−1)/2 and x=(m+1)/2, on each side of the center of the x direction beam along y=n/2. Whence, from Eqs. (3h) and (3n), $Y_{Cy} = 33 P_{Cy} a^3 / 2EI =$ and $Y_{Cx} = 11 P_{Cx} a^3 / 6EI$ respectively. Eq. (3p) upon substitution yields:

$$Y_C = \left(\frac{33 P_C a^3}{5EI}\right)\left(\frac{4}{2}\right) - \left(\frac{11 P_C}{10}\right)\frac{a^3}{EI} = \frac{121 P_C a^3}{10EI} \tag{3q}$$

3. THE PERFORMANCE CONTROL CUTVE

The two distinct transverse displacement values, Y_Y at P_Y and Y_C at P_C provide sufficient data for the construction of an accurate, bilinear load-displacement (push-down) curve for the class of regular grids addressed in this work, Figure. 4. The tri-linear curve $0YCP$, including the commonly accepted ductility limit Y_P at P, represents the complete, mathematically derived load displacement relationship for the example grillage of section 2.5. It is apparent that for the particular example, $Y_C \approx 1.54Y_Y$, and that all three points Y, C and P represent legitimate design limits as well as reliable PC criteria. However, the approximate, bilinear curve $0Y'CP$, can also be used for equally reliable design and/or investigative purposes, without resorting to complicated analysis for intermediate values of $Y_Y < Y < Y_P$, corresponding to $P_Y < P < P_C$. While segments $0Y'$, $Y'C$ and CP of the PC curves may be associated with allowable stress (ASD), load and resistance factor (LRFD) and plastic design (PD) philosophies respectively, the same segments may also be used to estimate percentage damage, assess global integrity or to propose intermediate control criteria, such as restricting maximum displacements to $(L_{\max.}/360)$ at 50% first yield or to $(L_{\max.}/240)$ at 75% of the collapse load. Any design based on Y_C and P_C might still be conservative due to basic assumptions presented under section 1.1 and that Y_P could be several times larger than Y_C.
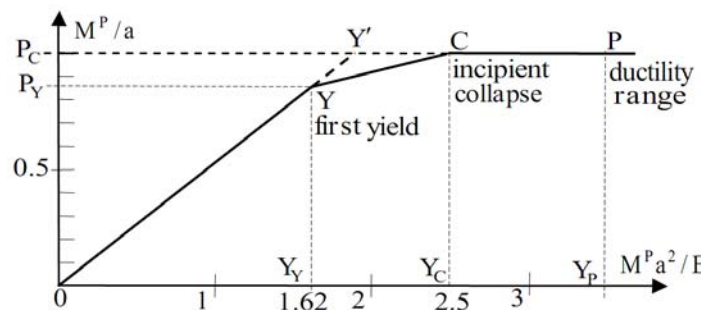

Figure 4. Load-displacement (push-down) curves for example grillage, section 2.5

It may be deducted from Figure. 4 that even for sparsely meshed, small gridworks, a two point bilinear plot can represent the response history of the system to a good degree of accuracy. The energy absorption capacities of the exact and approximate plots, signified by the areas under each curve are sufficiently close to suggest that the proposed plot is sufficiently accurate for all practical design purposes.

4. CONCLUSION

While the elastic analysis of grillages has received considerable attention in the past, their complete plastic treatment including the determination of displacements at incipient collapse has remained a challenging proposal until recent years.

A new, closed-form, exact formula for the computation of maximum transverse displacements of regular, rectangular, twistless grillages under a uniform distribution of normal nodal forces at incipient collapse has been introduced.

The applications of the proposed solutions have been demonstrated through simple generic examples. In fact a simple solution to a rather complex problem has been found. The determination of maximum plastic displacements of the class of grids discussed in the paper has been reduced to the summation of maximum displacements of two simple beams. The simplicity of the proposed solutions may be attributed to the regular formation of the gridwork that makes the analysis conducive to finite difference treatment. The proposed formulae are entirely suitable for manual as well as spreadsheet computations. The proposed designs satisfy all conditions of the uniqueness theorem, and as such, cannot be far from minimum weight solutions. However, they are limited to regular grid systems with members possessing no torsional resistance and meeting the boundaries at right angles.

The availability of reliable displacement values at first yield and incipient collapse provides the necessary data for performance control (PC) of regular grillages, interconnected truss systems and similar structures. It is hoped that PC will be recognized as a useful design method for ductile space frames and similar structures.

The use of the present methodology can be extended to study the maximum plastic displacements of similar gridworks with different combinations of free, fixed and hinged boundary support conditions. Given the availability of high powered means of computations, there is no reason why PC philosophies as described in this work, can not be extended to all types of space structures with complex geometries and non-uniform loading.

REFERENCES

1. Hayman J. The limit design of a transversely loaded square grid. *Journal of Applied Mechanics, ASME* **74**(1952) 153-539.
2. Hayman J. The plastic design of grillages, *Engineering*. **176**(1953) 804-911.
3. Hendry AW, Jaeger LG. A general method for the analysis of grid frameworks. *Proceedings Institution of Civil Engineers*. Part 3 (1953) 4-14.
4. Hodge PG. *Plastic Analysis of Structures*, McGraw Hill, New York, (1959) 75-79.
5. Holmes M. Plastic collapse of beam grillages, *Civil Engineering*. **56**(1961) 611-3.
6. Hongladaromp T, Rossow EC, Lee SL. Analysis of elastic-plastic grid systems, *Journal of Engineering Mechanics, Division ASCE*, No. EM1, **94**(1968) 241-71.
7. Grigorian M, Plastic design of regular orthotropic grids with fixed supports. *International Journal Mechaniasal Sciences*, **14** (1972) 197-203.
8. Marsh C. Collapse of point supported square grillages, *Journal of Structural Division, ASCE*, No. ST9, **103** (1973) 1703-12.
9. Stevens LK. Plastic design and space frames. *Engineering Plasticity*. Cambridge University Press, (1968) 627-33.
10. Schmidth LC. Alternative design methods for parallel-chord space trusses. *Structural Engineering*, No. 8, **50** (1972) 295-301.
11. Grigorian M, Lashgari M. Plastic design of trussed grids. *International Journal of Mechanical Sciences*. **15** (1973) 573-81.

12. Grigorian M. On diagrids of vierendeel frames. Proceedings*, IASS Pacific Symposium*. No. 7, **3** (1972) 617-625, Tokyo, Japan.
13. Grigorian M. An introduction to performance control for moment frames of uniform response under lateral loading. *Asian Journal of Civil Engineering (BHRC)*, No. 1, **14** (2013) 123-43.
14. Beedle LS. *Plastic Design of Steel Frames*. John Wiley & Sons (1958).
15. Heyman J. On the estimation of deflections in elastic plane structures. *Proceedins of the Institution of Civil Engineers*. **19** (1961) 39-60.
16. Neal BG. *The Plastic Methods of Structural Analysis*. Chapman & Hall Ltd. 1963.
17. Heyman J. *Plastic Design of Frames*. Cambridge University Press, 1973.
18. Scholtz H. Simplified interaction method for sway frames. *Journal of Structural Division*. ASCE, No. ST5, 110 (1984) 992-1007.
19. Grigorian M, Grigorian C. Recent developments in plastic design analysis of steel moment frames, *Journal of Constructional Steel Research*, **76** (2012)83-93.
20. Grigorian M, Grigorian C. Lateral Displacements of Moment Frames at incipient collapse, *Journal of Engineering Structures* **44** (2012) 174–85.
21. Grigorian M, Grigorian C. An Introduction to the Methodology of Earthquake Resistant Structures of Uniform Response. *Buildings* **2** (2012) 107-25.
22. Grigorian M, Grigorian C. Performance control for seismic design of moment frames, *Journal of Constructional Steel Research*, **67**(2011) 1106-14.
23. Grigorian M, Grigorian C. Performance control: a new elastic-plastic design procedure for earthquake resisting moment frames. *Journal of Structural Engineering Division, ASCE*, No. 6, **138** (2012) 473-83.
24. Grigorian M, Grigorian C. A new performance based design approach for moment resisting frames, *Canadian Journal of Civil Engineering*, **39** (2012) 1-11.
25. Priestley MJN, Kowalsky MJ. Displacement Based Seismic Design of Structures, IUSS Press, 2006.
26. Goel SC, Chao SH. Performance-Based Plastic Design, ICC, 2008.
27. Renton JD. A finite difference analysis of the flexural-torsional behaviour of grillages. *International Journal of Mechanical Sciences*. No. 3. **6**(1964) 209-24.
28. Wah T. Elasto-plastic Analysis of Uniform Grillages under Lateral Load, *Journal of Ship Building Research*, No.2, **13** (1969) 103-11.
29. Jones LL, Wood RH. Yield line analysis of slabs, Chatto and Windus, 1967.
30. Johansen KW, Yield line formulae for slabs, *Cement and Concrete Association*, 1972.
31. Faulks J. Minimum weight design and the theory of plastic collapse. *Quarterly of Applied Mechanics*. **10** (1954) 347-59.
32. Faulks J. The minimum weight design of moment frames. *Proceedings, Royal Society*. **A**.**223** (1954) 482-94.
33. Grigorian M. New theorem for the limit design of regular grids under continuousloading. Proceedings, 4th, Australian conf. on the mechanics of structures and materials. University of Queensland, Brisbane. pp. 387-399, 1973.
34. Grigorian M, Yaghmai S. A theorem for the plastic design of regular twistless grids under continuous transverse loading. *International Journal of Solids and Structures*. **9** (1973) 1421-30.

35. Grigorian M. A lower bound solution to the collapse of uniform rectangular grids on simple supports, *International Journal of Mechanical Sciences*. **13** (1971) 755-61.

36. Grigorian M. On efficient design of earthquake resistant moment frames, *Asian Joirnal of Civil Engineering (BHRC)*, No. 2, **14** (2013) 319-38.

APPENDIX A- THE CONSTITUTIVE EQUATION

Consider the elastic displacements, Y_{xy}, of the regular grillage of Figure.1a under normal nodal forces P_{xy}. Let I and J represent the sectional inertias of the x and y direction beams respectively. $D_x = EI / a^3$ and $D_y = EJ / b^3$ are the corresponding rigidities of the two sets of intersecting beams and $\rho = D_y / D_x$. The constitutive difference equation of bending of regular, flat, twistless grids may be presented [27,28] in matrix form as ;

$$
\begin{bmatrix}
12\left[\dfrac{D_x}{a^3}(\Delta_x - \nabla_x) + \dfrac{D_y}{b^3}(\Delta_Y - \nabla_Y)\right] & -\dfrac{6D_x}{a^2}(\Delta_x + \nabla_x) & -\dfrac{6D_y}{b^2}(\Delta_y + \nabla_y) \\
-\dfrac{6D_x}{a^2}(\Delta_x + \nabla_x) & \dfrac{2D_x}{a^2}(6 + \Delta_x - \nabla_x) & 0 \\
-\dfrac{6D_y}{b^2}(\Delta_y + \nabla_y) & 0 & \dfrac{2D_x}{a^2}(6 + \Delta_x - \nabla_x)
\end{bmatrix}
\begin{bmatrix} Y_{xy} \\ \phi_{xy} \\ \psi_{xy} \end{bmatrix}
=
\begin{bmatrix} P_{xy} \\ 0 \\ 0 \end{bmatrix}
\tag{A1}
$$

where, $\Delta_x = E_x^{+1} - 1$ and $\nabla_x = 1 - E_x^{-1}$ are the finite difference forward and backward shift operators, respectively. $\Delta_x - \nabla_x = E_x^{+1} - 1 + E_x^{-1} \equiv a^2 \dfrac{\partial^2}{\partial x^2}$ is the symmetric central difference operator that can also be identified as the second derivative of a continuous function with real values at equal intervals a. The finite difference operator $E_x^{\pm}$ performs the operation $E_x^{\pm} F(x) = F(x \pm 1)$ on any function of the variable x. The solution to Eq. (A1) for a simply supported grillage under a uniform distribution of normal nodal forces can be expressed as [12, 27 and 28];

$$
Y_{xy} = \frac{1}{12D_x} \sum_{i=1,3}^{m-1} \sum_{j=1,3}^{n-1} \frac{Q_{ij}}{L_{ij}} \sin \alpha x \cdot \sin \beta y, \quad \alpha = i\pi / ma \quad \text{and} \quad \beta = j\pi / nb
\tag{A2}
$$

$$
M_{xy} = \frac{1}{2} \sum_{i=1,3}^{m-1} \sum_{j=1,3}^{n-1} \frac{Q_{ij}}{L_{ij}} \left[\frac{\sin^2 \alpha}{2 + \cos \alpha} + \cos \alpha - 1 \right] \sin \alpha x \cdot \sin \beta y
\tag{A3}
$$

$$
N_{xy} = \frac{1}{2} \sum_{i=1,3}^{m-1} \sum_{j=1,3}^{n-1} \frac{Q_{ij}}{L_{ij}} \left[\frac{\sin^2 \beta}{2 + \cos \beta} + \cos \beta - 1 \right] \sin \alpha x \cdot \sin \beta y
\tag{A4}
$$

where,

$$Q_{ij} = \frac{4P_Y}{mn} \cot\frac{\alpha}{2} \cdot \cot\frac{\beta}{2} \tag{A5}$$

and,

$$L_{ij} = 2(1-\cos\alpha) + 2\rho(1-\cos\beta) - \frac{3\sin^2\alpha}{2+\cos\alpha} - \frac{3\sin^2\beta}{2+\cos\beta} \tag{A6}$$

APPENDIX B- INTER-NODAL DISPLACEMENTS

The additional maximum inter nodal displacement, $Y_{Inter.}$, of an x-direction beam with even number of normal nodal forces, i.e., m=odd, between nodes $(n-1)/2$ and $(n+1)/2$, as shown in Figure.5, may be expressed as;

$$Y_{Inter.} = Ma^2/4EI \qquad \text{and} \qquad Y_{Inter.} = M^P a^2/4EI = P_x a^3(m^2-1)/32EI \tag{B1}$$

Before first yield and after first yield respectively. From the design example 3, section 2.6 above;

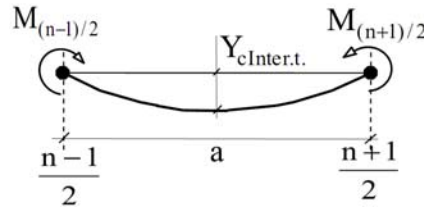Figure 5. Inter nodal displacement for beams with m bieng odd

$M^P = 9P_C a/5$, and $Y_C = 121P_C a^3/10EI$, compared with $Y_{Inter.} = 9P_C a^2/20EI$, i.e. $Y_{Inter.} = 9Y_C/242$, or less than 4% of the maximum nodal displacement ay incipient collapse.

APPENDIX C- BASIC STEPS

The basic steps involved in the computation of Y_{Cx} (for m=even) may be presented as
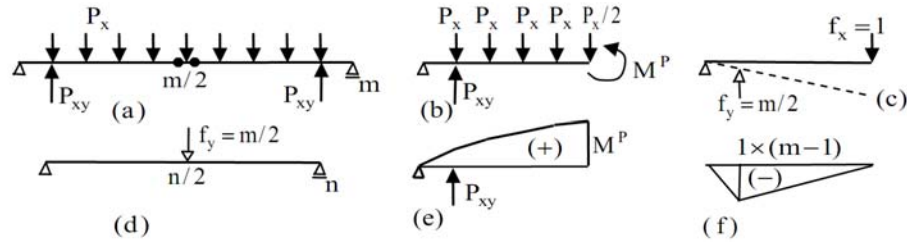
follows.

Figure 6. Displacement compatibility between first and last failing beams

(a)-First failing x-direction beam at $y=nb/2$, loading and interactive reactions P_{xy} at collapse. (b) Free body diagram of first failing beam along $y=nb/2$. (c) Same beam under virtual unit load $f_x=1$ at point of maximum deflection and its effect $f_y=m/2$. (d) Effect of unit load $f_x=1$ on the last stable y-direction beam, (e) bending moment distribution corresponding to (b). (f) Bending moment distribution corresponding to (c). Obviously, a similar set of steps can be presented for m as an odd integer.