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THE SCALED BOUNDARY SPECTRAL ELEMENT METHOD FOR DYNAMICS OF UNBOUNDED MEDIUMS WITH NON-HOMOGENEITY IN RADIAL DIRECTION

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ABSTRACT

Frequency domain procedure of the scaled boundary method (SBM) for analyzing unbounded mediums has been modified recently using the hybrid spectral element (SE) approach and lumped coefficient matrices. In this paper, the modified method is extended to analyze 2D non-homogenous unbounded domains. In the modified frequency domain approach, elasticity modulus and mass density of unbounded domains can be considered as power functions in the radial direction. Accuracy and efficiency of the presented method is evaluated by some benchmark examples and it is shown that the modified method leads to correct answers.

Keywords: Scaled boundary method; diagonal matrices; dynamic stiffness; unbounded domains; non-homogenous.

1. INTRODUCTION

In recent years, the boundary element method (BEM) has been developed for analyzing different engineering problems. For example, the boundary element method was applied to analyze seepage problems [1, 2] static problems [3, 4] and dynamic problems [5, 6]. The most attractive feature of the boundary element method is the mesh reduction property. By using the boundary element approach, dimension of problem reduces by one. Needing to a fundamental solution is the main drawback of the boundary element method. Fundamental solution is an analytical answer, which must satisfy the governed equations in the domain of the problem [7]. Recently a fundamental solution less boundary element method has been developed by Wolf and Song [8]. This method, which is named scaled boundary, is a semi analytical method and couples the advantages of the two mostly used finite element (FE) and boundary element methods [9]. The scaled boundary method (SBM) uses a scale center (SC) and two dimensionless local coordinates (η, ξ) for two-dimensional (2D) problems. Scale center must be selected in a manner

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that whole boundary can be viewed from it [7]. For bounded domains, dimension less radial coordinate (ξ) begins from the scaling center with the value zero.

This coordinate is selected equal to one on the boundary and is specified by $1 \leq \xi < \infty$ for unbounded domains [7]. The dimensionless circumferential coordinate (η) is chosen like the standard boundary elements $-1 \leq \eta < 1$. Domain discretization scheme of the scaled boundary method is shown in Fig. 1.

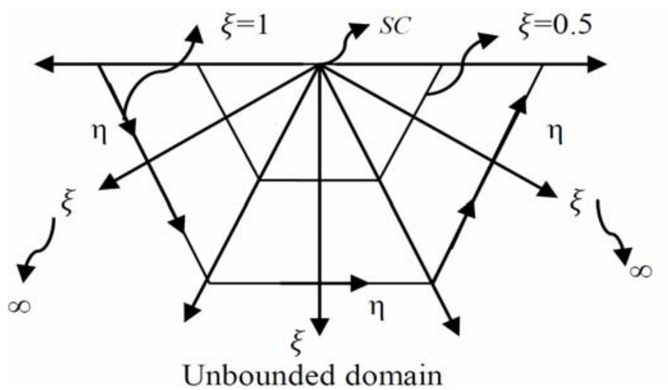


Figure 1. Domain discretization scheme in the scaled boundary method

Like the boundary element method, scaled boundary approach has been applied to solve different problems. For example, the method was used to analyze seepage problems [10, 11], static problems [12, 13] and dynamic problems [14, 15]. By coupling the scaled boundary method with other numerical approaches, the original method can be enhanced to obtain better features. For example, a more accurate stress field can be achieved by the hybrid mesh free-scaled boundary methods [16, 17]. Nonlinear behavior of near field media can be modeled using the coupled finite element method (FEM)-scaled boundary finite element method (SBFEM) [18].

Another hybrid approach can be achieved using the spectral element method. Spectral element is an *h-p* refined high order form of the finite element method [19] that can be used as a mass lumping procedure in the finite element analysis [20]. The conventional scaled boundary method has four coefficient matrices and as it is described in the next sections, two of the coefficient matrices can be lumped using the hybrid scaled boundary spectral element method (SBSEM). Scaled boundary method is a convenient way for analyzing unbounded mediums especially for dynamic problems. Radiation damping condition of unbounded mediums can be modeled using the scaled boundary method accurately. In the time domain procedure, calculating interaction force can be a very time consuming task. Required time of the analysis may be reduced by using the frequency domain approach [21]. In the most of the previously done researches with the scaled boundary method, homogenous unbounded mediums were analyzed, while limited numbers of the studies were dedicated for non-homogenous unbounded domains. Doherty and Deeks [22] investigated non-homogeneous unbounded mediums by the scaled boundary finite element method for static problems. Dynamic behavior of non-homogenous unbounded mediums was studied by Wolf [7] and Wolf and Song [8] where material properties were selected as power function of radial coordinate in the scaled boundary method. Bazyar and Song [23] developed the conventional frequency domain procedure of the scaled boundary finite element method for non-homogenous problems. They used material properties as power functions of both radial and circumferential coordinates.

The frequency domain procedure of the scaled boundary method leads to a system of ordinary differential equations [7]. For unbounded mediums, scaled boundary differential equation in dynamic stiffness can be solved as an initial value problem [8]. Therefore, an initial value is needed. This initial value can be achieved using a high frequency asymptotic expansion of dynamic stiffness matrix [7]. Recently a modified approach has been presented to expand dynamic stiffness matrix efficiently at high frequencies [21]. In the modified method, the hybrid scaled boundary spectral element approach was used to lump coefficient matrices. The enhanced method simplifies the original analyzing procedure by eliminating extra computational efforts. Formulation of the modified scaled boundary spectral element method was presented for homogenous unbounded domains [21]. In this paper, the proposed method is extended to analyze non-homogeneous unbounded mediums. The modified approach is employed a same technique which was presented by Bazyar and Song [23] however elasticity modulus and mass density of unbounded media are considered as power function of radial coordinate.

The paper is organized as follows: it commences with a brief review of the scaled boundary method, the modified method is then derived and two numerical examples are solved to evaluate accuracy and efficiency of the proposed approach.

2. SUMMARY OF THE SCALED BOUNDARY FINITE ELEMENT METHOD

The fundamental solution less, scaled boundary method is a semi analytical method which solves boundary value problems numerically in the circumferential direction and analytical in the radial direction [7]. The hybrid scaled boundary spectral element method can be used instead of the conventional method for enhancing the original features. Like the SBFEM, SBSEM has four coefficient matrices (E^0, E^1, E^2, and M^0). E^0 and M^0 are made only by shape functions while E^1 and E^2 contain derivatives of shape functions. E^0, E^1, E^2, and M^0 coefficient matrices can be achieved using Equation (1), Equation (2), Equation (3) and Equation (4) respectively.

$$[E^0] = \int_{-1}^{+1} [B^1]^T [D][B^1] |J| \, d\eta \tag{1}$$

$$[E^1] = \int_{-1}^{+1} [B^2]^T [D][B^1] |J| \, d\eta \tag{2}$$

$$[E^2] = \int_{-1}^{+1} [B^2]^T [D][B^2] |J| \, d\eta \tag{3}$$

$$[M^0] = \int_{-1}^{1} [N(\eta)]^T \rho [N(\eta)] |J| \, d\eta \tag{4}$$

In the equations, $[D]$ is the elasticity matrix for two-dimensional problems and $[\rho]$ is the density matrix. $[B^1]$ and $[B^2]$ are two important matrices in the scaled boundary method where $[B^1]$ is the first strain matrix and can be achieved using shape functions ($N(\eta)$). $[B^2]$ is the second strain matrix and contains derivatives of shape functions [7]. Like the mass matrix of bounded mediums, M^0 (can be considered as mass matrix of unbounded domains), can be lumped by using the spectral element idea. In this method, an appropriate integration rule (for example the Gauss-Lobatto-Legendre quadrature) must be used to integrate the coefficient matrices. In addition, shape functions must have the Kronecker delta function property. In the case of the E^0 matrix, a full diagonally lumped matrix can be obtained when a 1D boundary with constant x coordinate or y coordinate (dx=0 or dy=0) is evaluated [21]. In this paper, Lagrange polynomials are used as shape functions and Gauss-Lobatto-Legendre (GLL) quadrature is used to integrate the coefficient matrices. Lagrangian shape functions for 1D problem can be calculated using Equation (5).

$$N_i(\eta) = \prod_{k=1, k\neq i}^{p+1} \frac{\eta - \eta_k}{\eta_i - \eta_k} \tag{5}$$

For homogenous unbounded domains, dynamic stiffness matrix can be determined by the frequency domain approach of the scaled boundary method. The SBM leads to a system of ordinary differential equations. This first order nonlinear differential equation is introduced in Equation (6) [4].

$$([S^\infty(\omega)]+[E^1])[E^0]^{-1}([S^\infty(\omega)]+[E^1]^T)-(s-2)[S^\infty(\omega)]-\omega[S^\infty(\omega)]_{,\omega}-[E^2]+\omega^2[M^0]=0 \tag{6}$$

In the Equation (6), s is the dimension of the geometry (s=2 for 2D and s=3 for 3D problems). In this paper, only 2D problems are discussed hence s is considered equal to 2. $S^\infty(\omega)$ is the dynamic stiffness of homogenous unbounded domain. An initial condition is required to solve Equation (6) as an initial value problem. Previously, researchers proposed a high frequency asymptotic expansion for dynamic stiffness of unbounded mediums. This expanded matrix can be used as an initial condition for Equation (6). Equation (7) indicates this expanded matrix [6]:

$$[S^\infty(\omega)] = (i\omega)[C_\infty]+[K_\infty]+\sum_{j=1}^{m}[A_{(j)}](i\omega)^{-j} \tag{7}$$

Where, C^∞ and K^∞ are constant dashpot and spring matrices of unbounded media, respectively. For homogenous unbounded domains, unknown matrices of the Equation (7) can be obtained using the previously proposed modified SBSE method. In the modified method, additional computational efforts have been removed and an efficient formulation is available. Detailed description about the enhanced SBSE method is presented in the Ref. [21], however a comparison between the modified and original approaches can be reached by comparing flowchart of the methods. Fig. 2 presents flowchart of the original method and Fig. 3 introduced flowchart of the modified approach.

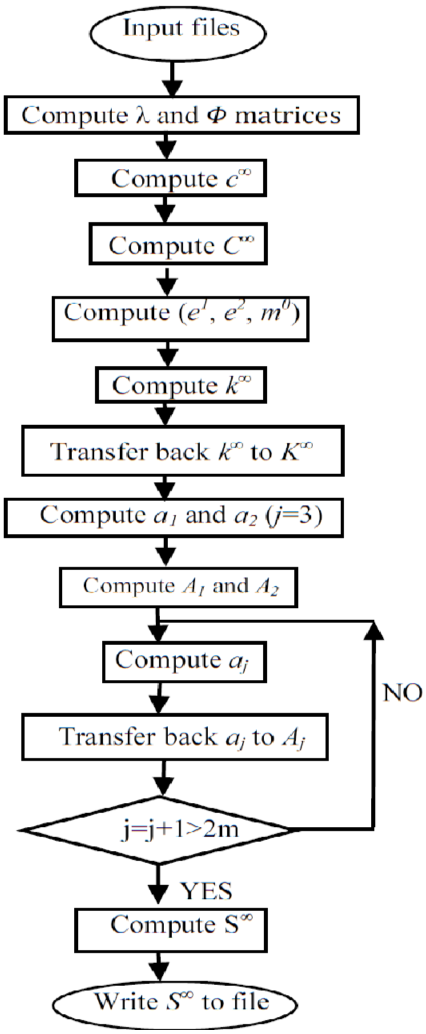


Figure 2. Flowchart of the SBFE (original) method to calculate $S^\infty(w)$ [21]

For non-homogenous unbounded domains, researchers presented scaled boundary differential equation in dynamic stiffness for different cases. Wolf [7] used material properties as power function of the radial coordinate while Bazyar and Song [23] improved the scaled boundary method for spatially nonhomogeneous materials. They used material properties as power functions of spatial coordinates [23]. For the unbounded domains with non-homogeneity in the radial direction, material properties can be stated as:

$$E(r) = \xi^{\alpha} E(\eta) \tag{8}$$

$$\rho(r) = \xi^{\beta} \rho(\eta) \tag{9}$$

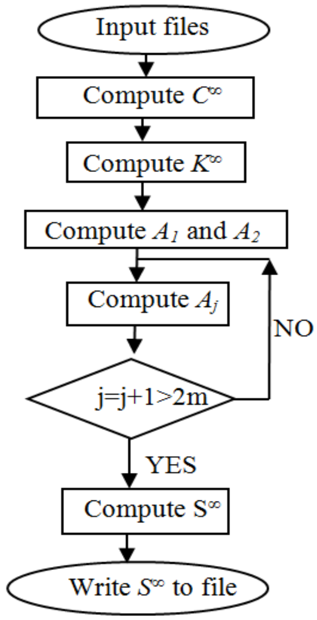


Figure 3. Flowchart of the modified SBSE method to calculate $S^\infty(w)$ [21].

Where values of $E(\eta)$ and $\rho(\eta)$ on the boundary (ξ=1) are defined as

$$E(\eta) = E_0 (\frac{r}{L})^{\alpha} \tag{10}$$

$$\rho(\eta) = \rho_0 (\frac{r}{L})^{\beta} \tag{11}$$

In these equations, α and β are non-homogeneity parameters of elasticity and density, respectively. E_0 and ρ_0 are the starting values for elasticity and density respectively. L is a characteristic length which could be chosen with different definitions. As Booker et al. [24] indicated, α can be selected with values between zero and one ($0 \leq \alpha \leq 1$) for geotechnical analysis. Required equations for determining dynamic stiffness matrix of non-homogenous unbounded mediums in the conventional scaled boundary finite element method were detailed by Bazyar and Song [23]. The main differences are in determining coefficient matrices (Equation (1)- Equation (4)) and in the scaled boundary differential equation (Equation (6)). In the next section, the scaled boundary method is modified for diagonal E^0 and M^0 coefficient matrices to analyse non-homogenous unbounded mediums.

3. THE MODIFIED METHOD FOR NON-HOMOGENOUS MEDIUMS

For non-homogenous unbounded mediums, elasticity and density matrices must be chosen as functions of radial coordinate. In the scaled boundary coordinates, elasticity and density matrices relate to circumferential and radial coordinates (Equation (8) and Equation (9)).

Since, coefficient matrices are integrated only over the circumferential direction (Equation (1)- Equation (4)), a separation of variables must be applied on elasticity and density matrices. Hence the resulted matrices can be defined as [23]:

$$[D(y)] = \xi^{\alpha}\,[D(\eta)] \tag{12}$$

$$\rho(y) = \xi^{\beta}\rho(\eta) \tag{13}$$

Differential equation for dynamic stiffness matrix of non-homogenous unbounded mediums has been derived previously and it is introduced in Equation (14) [23].

$$\begin{aligned}&([S^{\infty}(\omega)]+[E^{1}])[E^{0}]^{-1}([S^{\infty}(\omega)]+[E^{1}]^{T})\\&-(s+\alpha-2)[S^{\infty}(\omega)]-(1-0.5\alpha+0.5\beta)\,\omega[S^{\infty}(\omega)]_{,\omega}\\&-[E^{2}]+\omega^{2}[M^{0}]=0\end{aligned} \tag{14}$$

It must be mentioned that in the conventional scaled boundary method Equation (14) should be transformed in a new form. In the original approach, transformed coefficient matrices must be used. The transformed coefficient matrices in the original scaled boundary method can be achieved by Equation (15) - Equation (18).

$$[e^{0}]=[\Phi]^{T}[E^{0}][\Phi] \tag{15}$$

$$[e^{1}]=[\Phi]^{T}[E^{1}][\Phi] \tag{16}$$

$$[e^{2}]=[\Phi]^{T}[E^{2}][\Phi] \tag{17}$$

$$[m^{0}]=[\Phi]^{T}[M^{0}][\Phi] \tag{18}$$

In these equations, Φ can be obtained by the Cholesky decomposition of $[E^{0}]^{-1}$ (Equation (19)).

$$[E^{0}]^{-1}=[\Phi]\,[\Phi]^{T} \tag{19}$$

All of the transformation efforts are removed in the modified scaled boundary spectral element method and coefficient matrices are used on their original forms [21]. To obtain the needed initial condition for the differential equation (Equation (14)), the unknown coefficient matrices of Equation (7) must be determined. Therefore, Equation (7) must be substituted in Equation (14). By rearranging the obtained equation in descending order of the power series $i\omega$, Equation (20) can be formed.

$$
\begin{aligned}
&(i\omega)^2\,([C_\infty]^2\,[E^0]^{-1} - [M^0]) + (i\omega)\,([\Lambda][K_\infty] + \\
&[K_\infty][\Lambda] - (1+0.5\alpha+0.5\beta)[C_\infty] + [\Lambda][E^1]^T + \\
&[E^1][\Lambda]) + (i\omega)^0\,([\Lambda][A_1] + [A_1][\Lambda] + \\
&[K_\infty][E^0]^{-1}[K_\infty] + [K_\infty][E^0]^{-1}[E^1]^T + \\
&[E^1][E^0]^{-1}[K_\infty] + [E^1][E^0]^{-1}[E^1]^T - [E^2] \\
&-\alpha[K_\infty]) + (i\omega)^{-1}\,([\Lambda][A_2] + [A_2][\Lambda] + \\
&[K_\infty][E^0]^{-1}[A_1] + [A_1][E^0]^{-1}[K_\infty] + [E^1][E^0]^{-1}[A_1] + \\
&[A_1][E^0]^{-1}[E^1]^T + (-1+1.5\alpha-0.5\beta)[A_1]) \approx 0
\end{aligned}
\tag{20}
$$

In Equation (20), [□] is defined as:

$$[\Lambda] = [C_\infty][E^0]^{-1} \tag{21}$$

Each term of Equation (20) must be set equal to zero. First term yields to an equation for constant dashpot matrix of the unbounded media (C_∞). Hence, C_∞ can be achieved using Equation (22).

$$[C_\infty] = \sqrt{[M^0][E^0]} \tag{22}$$

Since the M^0 and E^0 coefficient matrices are positive, and they have diagonally lumped form (in the SBSE approach), C^∞ will be a positive definite matrix with diagonally lumped entries [21]. By setting the second term of Equation (20) equal to zero, a linear algebraic equation for K^∞ can be achieved. The resulted equation can be formed as:

$$
\begin{aligned}
&\Lambda][K_\infty] + [K_\infty][\Lambda] - (1+0.5\alpha+0.5\beta)[C_\infty] \\
&+[\Lambda][E^1]^T + [E^1][\Lambda] = 0
\end{aligned}
\tag{23}
$$

Equation (23) is a Lyapunov equation and can be solved using an iterative procedure. Third term of Equation (20) yields to a Lyapunov equation for the unknown coefficient matrix A_1. The resulted equation is shown in Equation (24). A same equation has been achieved for homogenous unbounded mediums, previously.

$$
\begin{aligned}
&[\Lambda][A_1] + [A_1][\Lambda] + [K_\infty][E^0]^{-1}[K_\infty] + \\
&[K_\infty][E^0]^{-1}[E^1]^T + [E^1][E^0]^{-1}[K_\infty] \\
&+[E^1][E^0]^{-1}[E^1]^T - [E^2] - \alpha[K_\infty] = 0
\end{aligned}
\tag{24}
$$

To calculate the unknown coefficient matrix A_2, the fourth term of Equation (20) must be set equal to zero. Equation (25) shows the resulted Lyapunov equation for A_2.

$$
\begin{aligned}
&[\Lambda][A_2]+[A_2][\Lambda]+[K_\infty][E^0]^{-1}[A_1]+\\
&[A_1][E^0]^{-1}[K_\infty]+[E^1][E^0]^{-1}[A_1]+\\
&[A_1][E^0]^{-1}[E^1]^T-(-1+1.5\alpha-0.5\beta)[A_1]=0
\end{aligned} \tag{25}
$$

If m is selected larger than two in Equation (7), unknown coefficient matrices A_{i+1} can be calculated using Equation (26).

$$
\begin{aligned}
&[\Lambda][A_{(i+1)}]+[A_{(i+1)}][\Lambda]+[K_\infty][E^0]^{-1}[A_{(i)}]+\\
&[A_{(i)}][E^0]^{-1}[K_\infty]+[E^1][E^0]^{-1}[A_{(i)}]+\\
&[A_{(i)}][E^0]^{-1}[E^1]^T-(-i+(1+0.5i)\alpha-0.5i\beta)[A_{(i)}]+\\
&\sum_{j=1}^{i-1}[A_{(j)}][E^0]^{-1}[A_{(i-j)}]=0
\end{aligned} \tag{26}
$$

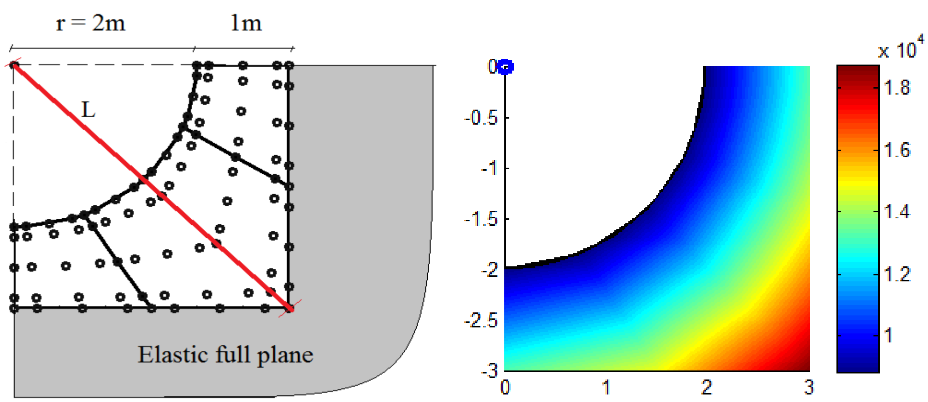
In the original form of Equation (26) (in the conventional SBFEM) the unknown matrices A_j must be calculated in the transferred form (a_j) then an inverse transformation should be applied for each i (i=1, 2,…, 2×m) while in the modified formulation, all these extra computational efforts are eliminated (see Fig. 3) [21]. Since all the unknown matrices of Equation (7) are determined, high frequency asymptotic expansion of dynamic stiffness matrix can be calculated. It is notable that the new formulation is only valid for the scaled boundary spectral element method with diagonally lumped E^0 and M^0 matrices [21]. To show differences between the modified methods for analysing homogenous and non-homogenous unbounded domains, it must be mentioned that some terms (which contain α and β) are added in Equation (23), Equation (24), Equation (25) and Equation (26) while other equations (Equation (21), Equation (22)) remain without any change.

4. NUMERICAL EXAMPLES

In this section, two different numerical examples are solved to assess accuracy and efficiency of the modified method. To achieve time history of displacements the inverse fast Fourier transformation is used in the examples. In these analyses, the extended mesh method (EMM) is used to verify obtained results. In addition, for the case α=0 and β=0 (homogenous media) achieved answers are compared with the results in the literature.

4.1 Circular Cavity Embedded in Full Plane

For the first example, a circular cavity embedded in an elastic full plane is selected. The plain strain condition is considered where wall of the cavity is subjected to a uniform pressure. In this example, radius of the cavity is selected equal to 2*m*. Young's (elasticity) modulus of the cavity is considered as a function of radius, which is introduced by Equation (10). The characteristic length (L) is selected equal to $18^{\,0.5}$ (see Fig. 4. (a)) . The initial value of elasticity modulus (E_0) is set equal to E_0=18720 kN/m^2. Other mechanical properties are considered constant, Poisson's ratio ν=0.3 and mass density $\rho=2\times10^3$ kg/m^3. As the problem is symmetrical, only one quarter of the cavity is analyzed. Fig. (4.a) presents considered geometry for cavity and its surrounded area. Non-homogeneity of the problem can be found in Fig. 4. (b) for the case that α is selected equal to one (α=1). In the analysis, Four twenty-five nodded spectral elements and four five-nodded scaled boundary spectral elements are used to model near field bounded and far field unbounded mediums, respectively. Sub-structure method is undertaken to achieve lumped E^0 and M^0 coefficient matrices of unbounded and mass matrix of bounded mediums.



(a) (b)

Figure 4. (a) geometry discretization for a circular cavity embedded in full plane using SE-SBSEM (b) Distribution of elasticity modulus in the geometry of problem for α=1 (circular cavity embedded in full plane)

The used extended mesh for this study is shown in Fig. 5.

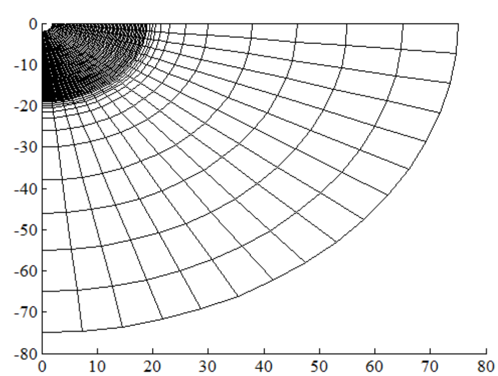


Figure 5. Extended mesh for modeling a circular cavity embedded in full plane

In the extended mesh method, 1375 nodes are used while in the coupled scaled boundary method, total number of nodes is equal to 85. Time history of the applied uniform pressure is considered as a triangular function, which is defined in Fig. 6(a). Duration of the load is 0.1*s* and the peak value of the load is selected equal to 10*kN*. Amplitude of the load function in the frequency domain is plotted in Fig. 6(b).

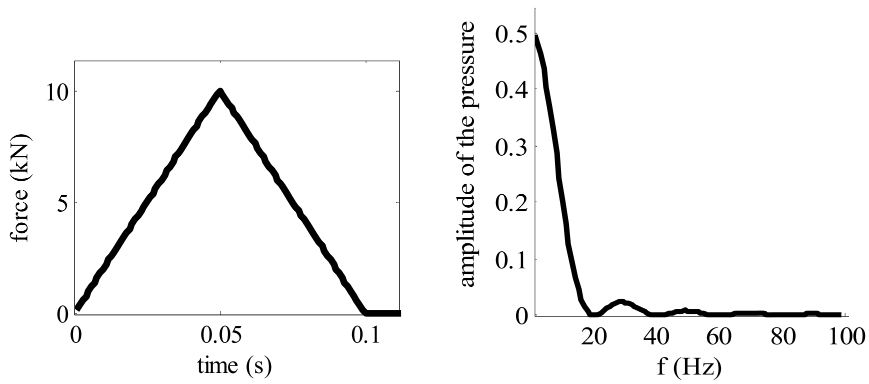


Figure 6. (a) Time history of considered triangular load function (b) Amplitude of the pressure in frequency domain

Firstly, homogenous case (α=0) is investigated. In this analysis, four different methods are used. Analytical solution of this problem is in available; for example can be found in Ref. [26]. Obtained results are plotted in Fig. 7. It is shown that, the modified scaled boundary method leads to an excellent agreement with the EMM and the original SBM. In addition, a good agreement with the analytical solution is achieved. It must be mentioned that the run time of the modified SBSE method is equal to 10.87*s* (in a conventional laptop) while the extended mesh method needs to 62.13*s* for analyzing the considered problem.

For investigating the accuracy of the modified SBSE approach in the non-homogenous mediums, two different cases are considered. For the case α=1, variability of elasticity modulus by radial coordinate can be found in Fig. 8.

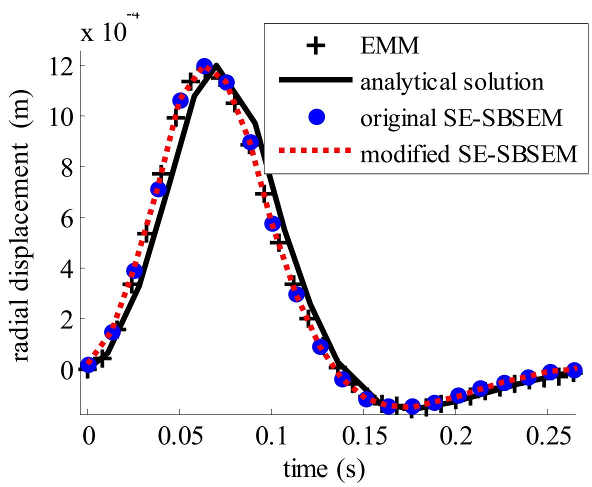


Figure 7. Radial displacement time history of cavity wall (α=0)

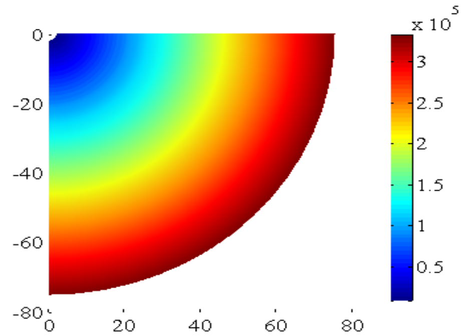
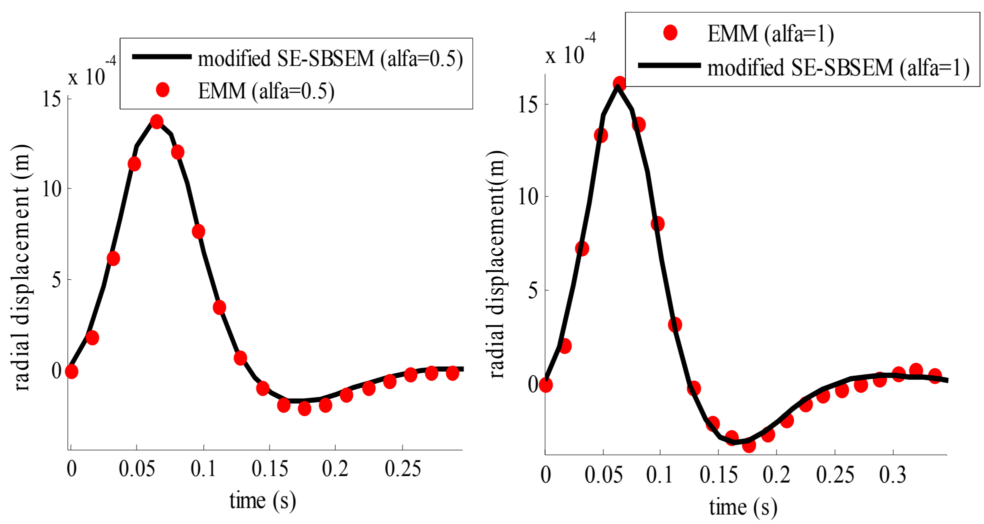


Figure 8. Distribution of elasticity modulus in the extended mesh method for α=1 (circular cavity embedded in full plane)

Obtained radial displacement time histories of the cavity wall for α=1 and α=0.5 are plotted in Fig. 9. As this figure shows, excellent agreement between the EMM and the modified SBSE approach is achieved. This figure also shows that by increasing the parameter α, an increment in the peak of the displacements is occurred.

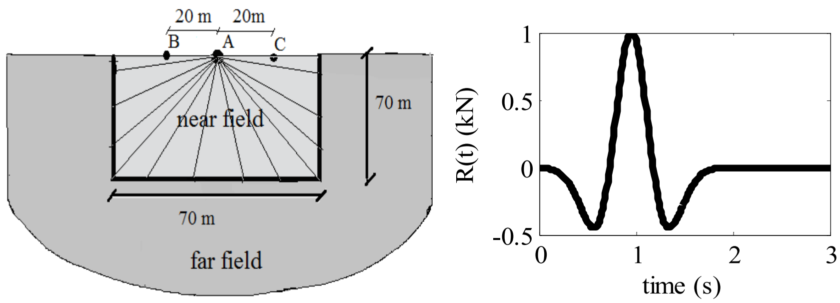


(a) (b)

Figure 9. Radial displacement time history of cavity wall for (a). α=0.5 (b) α=1

4.2 Elastic Half Plane Subjected to a Dynamic Load

To assess efficiency of the proposed procedure, an elastic half plane, which is shown in Fig. 10(a), is considered. The considered half plane is subjected to a Ricker wavelet type horizontal dynamic load function. In this example, Initial elasticity modulus of half plane E_0=2.66×10^5 kN/m^2, constant Poisson's ratio ν=0.33 and constant mass density ρ=2×10^3 kg/m^3 are selected to perform the analysis. Dynamic load is applied at the point *B* (Fig. 10(a)) with coordinates *x*=-20 and y=0. Time history of the load function is plotted in Fig. 10(b) where this function can be obtained using Equation (27) and frequency content of the load function can be achieved using Equation (28) [25].



(a) (b)

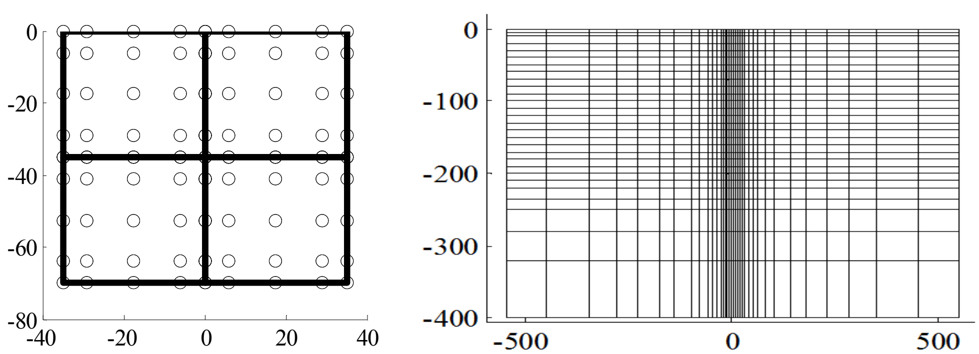
Figure 10. (a) Geometry of the problem (b) Time history of the load function

The known parameters of the load function are selected as A_R=1, t_0=1/π, t_s=3/π.

$$R(t) = A_R(1-2(\frac{t-t_S}{t_0})^2)\exp(-(\frac{t-t_S}{t_0})^2) \tag{27}$$

$$R(\omega) = .5\sqrt{\pi}A_R\,t_0\,(\omega t_0)^2\exp(-.25(\omega t_0)^2) \tag{28}$$

An extended mesh is generated for this problem, which discretize a domain with length and width equal to 1100*m* and 400*m*, respectively. Fig. 11 presents the used spectral element and extended meshed for this problem. In the SE-SBSE method, 81 nodes are used while 1073 nodes are employed in the EMM. Response of the domain are calculated for only homogenous cases (α=0 and β=0). Estorff and his coworker [27] was used the coupled finite element- boundary element method (FE-BEM) to analyze this problem (in homogenous condition).



(a) (b)

Figure 11. (a). Geometry discretization using spectral elements (b) extended mesh

Displacement time history of point *A* and point *B* are determined by three different methods. Fig. 12 shows the resulted answers. As this figure indicates, all methods converge to same answer, which means that the EMM and the modified SBSEM can able to model this problem accurately. By increasing period of the vibration, reflected waves can affect answer of the EMM while the SBSE method can be used for any considered period.

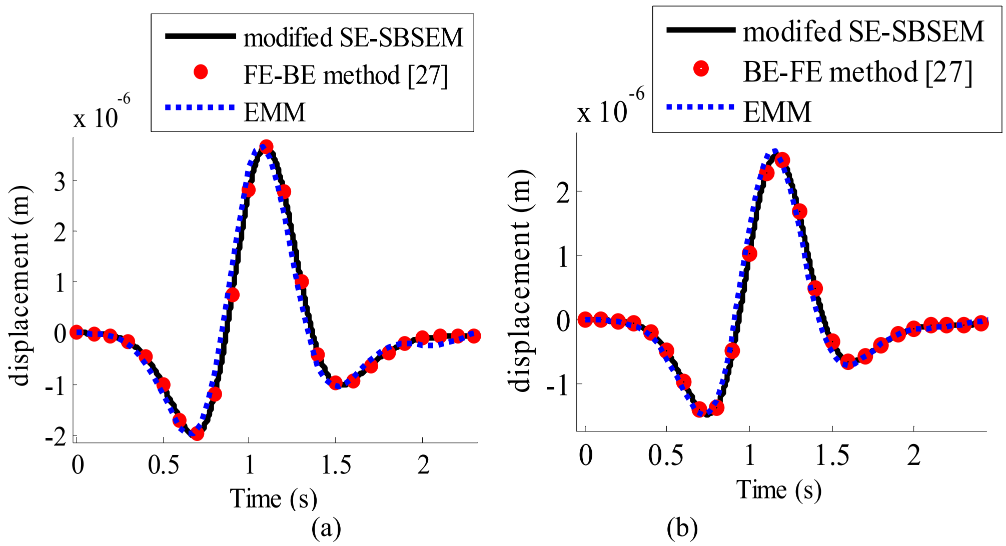


Figure 12. Horizontal displacement time histories of (a) point *A* (b) point *C*

5. CONCLUSION

In this paper, the frequency domain procedure of the modified scaled boundary spectral element method is extended to analyze non-homogenous unbounded mediums. The presented formulation is valid for non-homogeneity in spatial coordinates however only the non-homogeneity in radial direction is studied. By the proposed formulation some extra computational efforts of the original scaled boundary finite element method can be removed. Accuracy and efficiency of the discussed method is evaluated by solving two numerical examples. Five verifications are presented to ensure that the modified SBSEM leads to correct answers. Hence it can be concluded that the modified scaled boundary spectral element method can be used to analyze dynamics of non-homogenous unbounded mediums, efficiently.

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