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Technical Note

EVALUATION OF THE RADIUS OF CURVATURE BEAM EQUATION USING VARIATIONAL ITERATION METHOD

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ABSTRACT

An analytical technique is used to solve the radius of curvature beam equation called the variational iteration method (VIM). VIM leads to high exactness of the solutions with only one repetition. It has been found that VIM is very prolific, rapid, functional, and do not demand small perturbation and is also sufficiently accurate to both linear and nonlinear problems in engineering. The obtained consequences show that the approximate solution is uniformly legitimate on the whole solution field in comparison with the numerical solution. VIM could simply be enlarged to other powerfully nonlinear problems and it could be found widely feasible in engineering and science. The results of this method (VIM) are compared with the obtained results of numerical solution that shows the results of the present method are in excellent agreement with numerical solution.

Keywords: Radius of curvature beam equation; variational iteration method; numerical method (NM); nonlinear equation.

1. INTRODUCTION

Most of engineering problems are nonlinear and in most cases it is difficult to solve such equations, especially analytically. Perturbation method is one of the well-known methods to solve nonlinear problems; it is based on the existence of small/large parameters, the so-called perturbation quantity [1,2].

Recently, considerable attention has been paid towards approximate solutions for analytically solving nonlinear differential equation. Many nonlinear problems do not contain such perturbation quantity, so to overcome the shortcomings, many new techniques have appeared in open literature such as; Exp-function method [3,4], homotopy perturbation method (HPM) [5,6], Adomian decomposition method (ADM) [7,8], parameter–expansion

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method (PEM) [9,10], parameterized perturbation method (PPM) [11], Hamiltonian approach [12], homotopy analysis method (HAM) [13,14], Variational iteration method (VIM) [15,16,17,18,19,20] and differential transformation method (DTM) [21,22].

Among these methods, VIM is considered to solve the radius of curvature beam equation with considering nonlinear terms of the equation $\rho y'' - \left[1+(y')^2\right]^{\frac{3}{2}} = 0$ in this paper. It should be noted that Beer et al. [23] ignored the nonlinear part of the equation. Recently Samaee et al. [24] solved the equation with considering the nonlinear part of the equation using homotopy perturbation method and parameterized perturbation method.

This paper has been collocated as follows: first, we describe the basic concept of Variational Iteration Method, after which we consider the mathematical formulation of two examples which include a simply supported beam and a cantilever beam. Also, application of variational iteration method was studied to demonstrate the applicability and preciseness of this method. This was followed by a presentation of some comparisons between analytical and numerical solutions and then compare of achieved results with the solution of equation $\rho y'' - 1 = 0$ (in Mechanics of Materials [23] the nonlinear part is ignored). Eventually, we showed that VIM is an accurate and efficient method to solve neutral functional–differential equations.

2. BASIC CONCEPT OF VARIATIONAL ITERATION METHOD

The variational iteration method was first proposed in 1997 by He [15] and is a modified general Lagrang's multiplier method. To clarify the basic ideas of VIM, we consider the following differential equation:

$$Lu + Nu = g(t) \tag{1}$$

Where L is a linear operator, N a nonlinear operator and g(t) an inhomogeneous term.

According to VIM, we can write down a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\left(Lu_n(\tau) + F\tilde{u}(\tau) - g(\tau)\right)d\tau \tag{2}$$

Where λ is a general Lagrangian multiplier [16] which can be identified optimally via the variational theory. The subscript n indicates the nth approximation and $\tilde{u}$ is considered as a restricted variation [25], i.e., $\delta\,\tilde{u}_n = 0$.

3. SOME EXAMPLES

Now we apply the proposed technique to solve some nonlinear examples. In these examples which include a simply supported beam and a cantilever beam, we estimate the radius of curvature of two beams with considering nonlinear terms.

3.1 Example

Consider the direct beam AB under bending moment. The curvature in beam is caused by bending, making its displacement. The purpose of the beam displacement is the lateral displacement of the beam's central axis perpendicular to its original condition. Fig. 1 shows the value of *y*, (displacement of point C for beam AB). To obtain displacement of the beam, first, beam displacement equation is required. To do so, the relation of bending moment and radius of curvature is used [Eq. (3)]:

$$\frac{1}{\rho}=\frac{M}{EI} \tag{3}$$

Where Mis the bending moment, Ethe modulus of elasticity, ρ the radius of beam curvature and I the moment of inertia of the cross-section about its neutral axis. Now by using of curvature equation for function y:

$$\frac{1}{\rho}=\frac{y''}{\left(1+y'^2\right)^{\frac{3}{2}}} \tag{4}$$

From Eqs. (3) and (4) we have:

$$\frac{M}{EI}=\frac{y''}{\left(1+y'^2\right)^{\frac{3}{2}}} \tag{5}$$

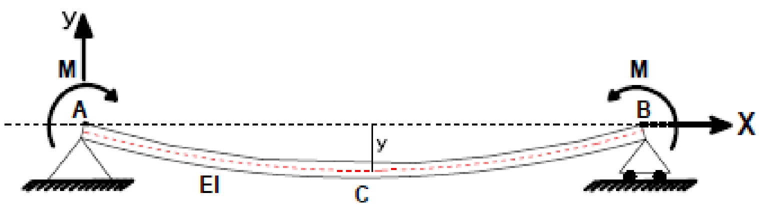

Figure 1. A schematic for the simply supported beam

The nonlinear radius of curvature beam equation is given by [Eq. (4) can be rewritten as follows]:

$$\rho y''-\left[1+\left(y'\right)^2\right]^{\frac{3}{2}}=0 \tag{6}$$

With the following initial conditions:

$$y\left(0\right)=0,\quad y'\left(0\right)=0,\quad \text{L=10.} \tag{7}$$

Where y is deflection, y' is the slope of the beam and ρ is radius of curvature beam.

3.1.1 Implementation of VIM

Nonlinear Eq. (6) can be expanded by means of Taylor expansion:

$$\rho y'' - 1 - \left(\frac{3}{2}\right)(y')^2 - \left(\frac{3}{8}\right)(y')^4 = 0 \tag{8}$$

In order to solve Eq. (8) using the VIM, we construct a correction functional as follows:

$$y_{n+1} = y_n + \int_0^x \left(\frac{2}{\rho}(\tau - x)\cdot\left(\frac{d^2 y_n}{d\tau^2} - \frac{1}{\rho} - \frac{3}{2\rho}\left(\frac{dy_n}{d\tau}\right)^2 - \frac{3}{8\rho}\cdot\left(\frac{dy_n}{d\tau}\right)^4\right)\right) d\tau \tag{9}$$

The first approximation of Eq. (8) is:

$$y_0 = \frac{x^2}{2\rho} - \frac{5x}{\rho} \tag{10}$$

Using the variational we have:

$$\begin{aligned} y_1 = {} & \frac{1}{2}\frac{x^2}{\rho} - \frac{5x}{\rho} - \frac{1}{8}\frac{x^6}{\rho^6} + \frac{2}{5}\left(\frac{3}{8}\frac{x}{\rho^5} + \frac{15}{2\rho^5}\right)\frac{x^5}{\rho} + \frac{1}{2}\left(-\frac{15}{2}\frac{x}{\rho^5} - \frac{3}{2\rho^3} - \frac{225}{4\rho^5}\right)\frac{x^4}{\rho} \\ & + \frac{2}{3}\left(-x\left(-\frac{3}{2\rho^3} - \frac{225}{4\rho^5}\right) + \frac{15}{\rho^3} + \frac{375}{2\rho^5}\right)\frac{x^3}{\rho} + \left(-x\left(\frac{15}{\rho^3} + \frac{375}{2\rho^5}\right) - \frac{75}{2\rho^3} - \frac{1875}{8\rho^5}\right)\frac{x^2}{\rho} \\ & - 2\left(-\frac{75}{2\rho^3} - \frac{1875}{8\rho^5}\right)\frac{x^2}{\rho} \end{aligned} \tag{11}$$

It has shownthe comparison of the obtained solutions of VIM and numerical at $\rho = 19$ in Fig. 2 and 3 dimensional VIM results have presented in Fig. 3.

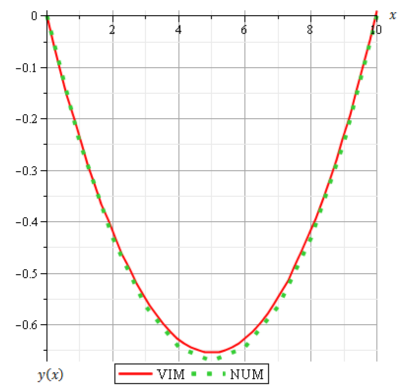

Figure 2. The comparison of the obtained solution with the numerical solution at $\rho = 19$

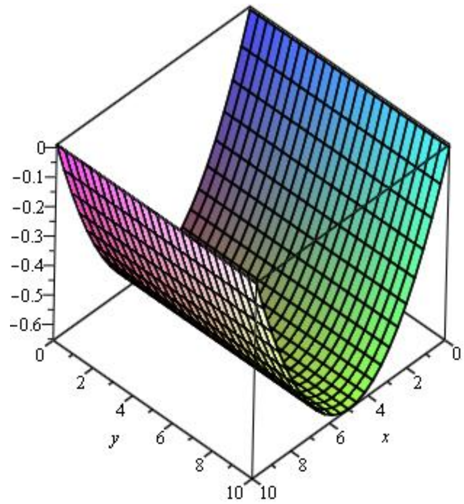

Figure 3. VIM solution at ρ = 19

Tables 1 and 2 andFig. 4 represent comparison of analytical solution of y based on displacement with the numerical solution at ρ = 19 , that in Table 2 and Fig. 4 we have considered that the nonlinear part of equation is ignored ($y' = 0$ in equation).

Table 1: Comparison between the results of VIM and numerical solution at ρ = 19

x	VIM	Numerical Solution	Error (|*VIM-NS*|)
1	-0.2365869655	-0.2438733110	0.0072863455
2	-0.4201658568	-0.4313608100	0.0111949532
3	-0.5509007159	-0.5641414070	0.0132406911
4	-0.6289077592	-0.6433637410	0.0144559818
5	-0.6542563338	-0.6696977700	0.0154414362
6	-0.6269694918	-0.6433637410	0.0163942492
7	-0.5470241810	-0.5641414070	0.0171172260
8	-0.4143510545	-0.4313608100	0.0170097555
9	-0.2288338957	-0.2438733110	0.0150394153
10	0.0096913372	0.0000000000	0.0096913372

Now by using of curvature equation for equation $\rho y'' - 1 = 0$ (in mechanics of materials [23] the nonlinear part is ignored), we obtain the results of table 2 and Fig.4.

Table 2: Comparison between the results of VIM and numerical solution at ρ = 19 for $y' = 0$

x	VIM	Numerical Solution	Error (|VIM-NS|)
1	-0.2368421053	-0.2368420000	0.0000001053
2	-0.4210526316	-0.4210520000	0.0000006316
3	-0.5526315789	-0.5526310000	0.0000005789
4	-0.6315789474	-0.6315790000	0.0000000526
5	-0.6578947368	-0.6578940000	0.0000007368

6	-0.6315789474	-0.6315790000	0.0000000526
7	-0.5526315789	-0.5526310000	0.0000005789
8	-0.4210526316	-0.4210520000	0.0000006316
9	-0.2368421053	-0.2368420000	0.0000001053
10	0.0000000000	0.0000000000	0.0000000000

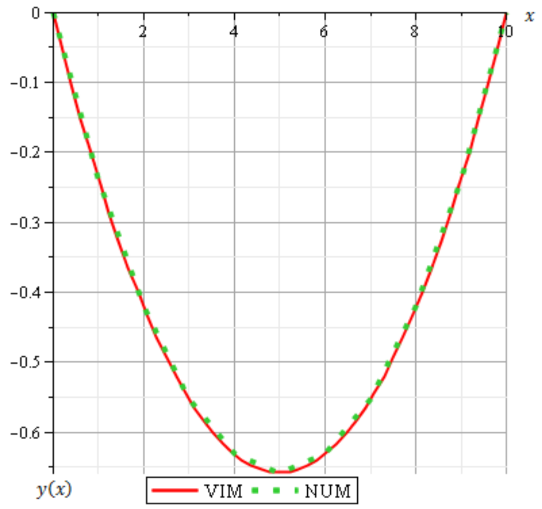

Figure 4. The comparison of the obtained solution with the numerical solution at ρ = 19 (For $y' = 0$ in equation $\rho y'' - \left[1+(y')^2\right]^{\frac{3}{2}} = 0$)

3.2 Example 2

Fig. 5 shows a cantilever beam. Consider the following nonlinear radius of curvature beam equation:

$$\rho y'' - \left[1+(y')^2\right]^{\frac{3}{2}} = 0 \tag{12}$$

With the following initial conditions:

$$y(0) = 0, \quad y'(0) = 0. \tag{13}$$

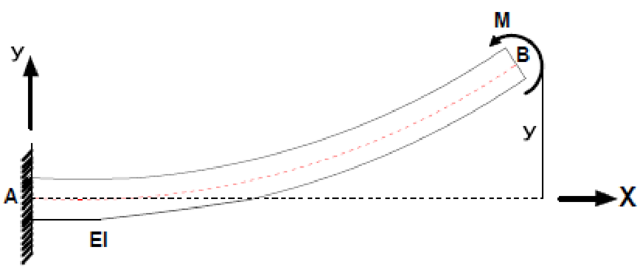

Figure 5. A schematic for the cantilever beam

3.2.1 Implementation of VIM

Nonlinear Eq. (12) be expanded by means of Taylor expansion:

$$\rho y'' - 1 - \left(\frac{3}{2}\right)(y')^2 - \left(\frac{3}{8}\right)(y')^4 = 0 \tag{14}$$

In order to solve Eq. (12) using the VIM, we construct a correction functional as follows:

$$y_{n+1} = y_n + \int_0^x \left(\frac{2}{\rho}(\tau - x) \cdot \left(\frac{d^2 y_n}{d\tau^2} - \frac{1}{\rho} - \frac{3}{2\rho}\left(\frac{dy_n}{d\tau}\right)^2 - \frac{3}{8\rho} \cdot \left(\frac{dy_n}{d\tau}\right)^4 \right) \right) d\tau \tag{15}$$

The first approximation of Eq. (15) is:

$$y_0 = \frac{x^2}{2\rho} \tag{16}$$

Using the variational we have:

$$y_1 = \frac{1}{2}\frac{x^2}{\rho} + \frac{1}{40}\frac{x^6}{\rho^6} + \frac{1}{8}\frac{x^4}{\rho^4} \tag{17}$$

Comparison of results for different parameters via numerical and VIM is presented in Fig. 6. Fig. 7 shows VIM results in three dimensional shape.

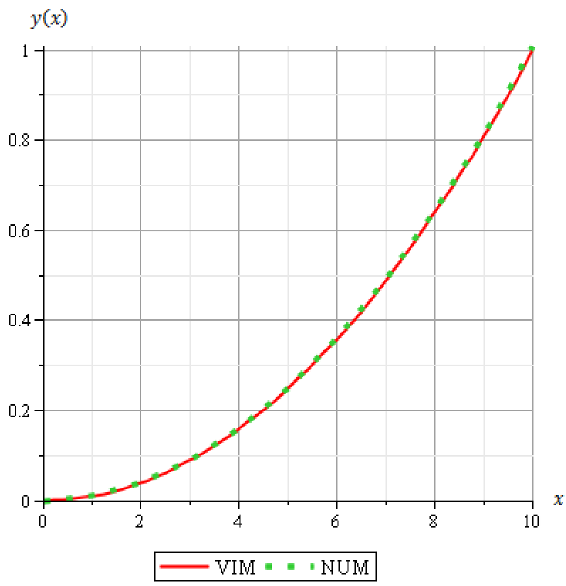

Figure 6. The comparison of the obtained solution with the numerical solution at $\rho = 50$

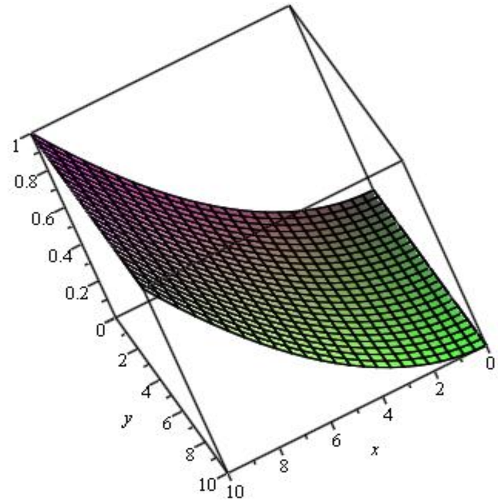

Figure 7. VIM solution at $\rho = 50$

Tables 3 and 4 represent comparison of analytical solution with the numerical solution for different parameters of the system.

Table 3: Comparison between the results of VIM and numerical solution at $x = 5$

ρ	VIM	Numerical Solution	Error (|VIM-NS|)
50.0	0.2500250250	0.2506281960	0.0006031710
50.1	0.2495258237	0.2501254280	0.0005996043
50.2	0.2490286124	0.2496246790	0.0005960666
50.3	0.2485333794	0.2486291850	0.0000958056
50.4	0.2480401125	0.2486291850	0.0005890725
50.5	0.2475488006	0.2481344180	0.0005856174
50.6	0.2475488006	0.2476416200	0.0000928194
50.7	0.2465719942	0.2471507810	0.0005787868
50.8	0.2460864768	0.2466618880	0.0005754112
50.9	0.2456028685	0.2461749300	0.0005720615

Table 4: Comparison between the results of VIM and numerical solution at $\rho = 50$

x	VIM	Numerical Solution	Error (|VIM-NS|)
1	0.0100000400	0.0100009980	0.0000009580
2	0.0400006401	0.0400159760	0.0000153359
3	0.0900032411	0.0900811050	0.0000778639
4	0.1600102466	0.1602568690	0.0002466224
5	0.2500250250	0.2506281960	0.0006031710
6	0.3600519146	0.3613053010	0.0012533864
7	0.4900962282	0.4924245930	0.0023283648
8	0.6401642594	0.6441493200	0.0039850606
9	0.8102632903	0.8166697250	0.0064064347
10	1.0004016000	1.0102052300	0.0098036300

Now if it consider that the nonlinear part of equation is ignored, table 5 and Fig. 8 can shows this assumption at ρ=50 (for $y'=0$ in equation $\rho y''-\left[1+\left(y'\right)^2\right]^{\frac{3}{2}}=0$).

Table 5: Comparison between the results of VIM and numerical solution at ρ = 50

x	VIM	Numerical Solution	Error (|VIM-NS|)
1	0.01	0.01	0.00
2	0.04	0.04	0.00
3	0.09	0.09	0.00
4	0.16	0.16	0.00
5	0.25	0.25	0.00
6	0.36	0.36	0.00
7	0.49	0.49	0.00
8	0.64	0.64	0.00
9	0.81	0.81	0.00
10	1.00	1.00	0.00

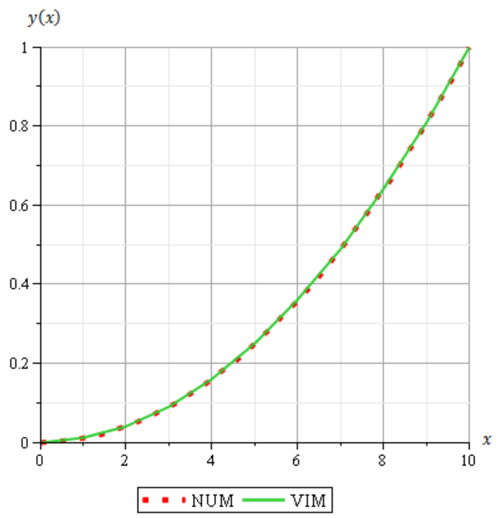

Figure 8. The comparison of the obtained solution with the numerical solution at ρ = 50(For $y'=0$ in equation $\rho y''-\left[1+\left(y'\right)^2\right]^{\frac{3}{2}}=0$)

4. CONCLUSION

In this paper, variational iteration method is applied to find the solution of radius of curvature beam equations. Comparison of the results of this method with the results of numerical solution reveals that the variational iteration method is very effective and convenient. In addition, we also have checked this problem with considering the nonlinear part of y' which this assumption is not a common and usual work in many problems in

Mechanics of Materials [23], because although it increases hardness and difficulty, it increases accuracy in answers too. These obtained results are more compatible with reality.

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