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APPLICATION OF SINGULAR VALUE DECOMPOSITION IN SYMMETRIC STRUCTURES BY FORCE METHOD

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ABSTRACT

Different methods are used for the formation of the null basis matrix of a structure followed by calculating the element forces when the force method of structural analysis is utilized. Singular value decomposition (SVD) method is one of the algebraic methods in force method that uses equilibrium matrix in its computation of finding element forces. By increasing the dimensions of the equilibrium matrix in large scale structures, the time needed for making orthogonal matrices grows. In recent decades many researchers have worked on block-diagonalization of structural matrices such as stiffness matrix in symmetric structures to reduce the computational time and simplify the equations. Block-diagonalization of equilibrium matrix can also reduce the computational time in algebraic force methods. In this paper block-diagonalization of equilibrium matrices of circulant structure is performed using Kronocker product and SVD method is utilized to calculate the block-diagonalization null basis matrix. Finally the results are compared to those of the method without block-diagonalization and other algebraic methods.

Keywords: Force method; null basis matrix; singular value decomposition; block-diagonalization of equilibrium matrix.

1. INTRODUCTION

Group representation theory has been applied to symmetric structures by many researchers. Block-diagonalizing of stiffness matrix by concept of structural symmetry properties is a way to reduce computation in structural analysis. Some examples of these can be found in Zhang and Chunhang [1], Dinkevich [2], Healey and Treacy [3], Zlokovich [4]. Kangwai et al. [5] in 1999 explained the results of these researchers briefly. Kaveh [6] used canonical

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forms of matrices by concept of matrix algebra, graph theory, linear algebra and group theory to solve the corresponding equations easily. Kaveh and Fazli [7] combined graph theory and group theoretical concepts for efficient eigensolution of adjacency matrices of graphs. Kaveh and Nikbakht [8] provided a mathematical foundation and a logical means to deal with symmetry instead of looking for different boundary conditions to be imposed for symmetric structures, as in the traditional methods. Graph products in regular structures was utilized by Kaveh et al. [9] to decompose matrices involved in dynamic equilibrium into the submatrices of lower dimensions. Kaveh [10] has reviewed the progress made in the analysis of structures by means of the force method. Zlokovich [11] has applied the effect of symmetry in structural analysis by force method. He used irreducible representation theory indirectly to consider the effect of symmetry on equilibrium relations and make a statically indeterminate structure to a determinate structure by a set of fully symmetric cuts. Bossavit [12] argued that group representation theory could apply to equilibrium and compatibility relations of symmetric structures. Kangwai and Guest [13] used group representation theory to block-diagonalize the equilibrium matrix of symmetric structures based on mathematical base described by Bossavit.

There is another way to block-diagonalize the stiffness matrix of specific type of symmetric structures. Stiffness matrix of circulant structures can be block-diagonalized with the aid of Kronocker product. Kaveh and Rahami [14] demonstrated how to block-diagonalize the stiffness matrix by concept of Kronocker product. In this method, first the diagonalizablity condition of ordinary and block matrices should be satisfied and then it is explained how to find a transformation matrix to make a diagonalized matrix. In matrix algebra, if eigenvectors make an identity orthogonal set, a square matrix can be made a diagonalized matrix by its eigenvalues. Hermitian matrices could be diagonalized and its diagonal contains the eigenvalues of the matrix. By using permutation matrix instead of Hermitian matrix, the matrix will be block-diagonalized instead of being diagonalized. In circulant truss structures which have rectangular equilibrium matrix with a regular pattern, the equilibrium matrix could be block-diagonalized proportional to number of rotations.

In structural analysis by force method, different methods are applied to calculate the element forces based on calculating the null basis matrix. Kaveh [15] considered the application of discrete mathematics rather than the more usual calculus-based methods of analysis of structures and finite element methods. Singular value decomposition, subset of algebraic force methods, works based on decomposition of equilibrium matrix. Pellegrino and Calladine [16] and Pellegrino [17] used singular value decomposition (SVD) of equilibrium matrix in structural analysis to prevent calculating the inversion of stiffness matrix. This method could exactly analyze the structures which contain instability. Rahami et al. [18,19] also has applied the singular value decomposition of the equilibrium matrix to analyze regular structures with node irregularity.

In this paper the block-diagonalized null basis of circulant truss structures is calculated by SVD after block-diagonalizing the equilibrium matrix by Kronocker product. Advantages and disadvantages of this method will be compared to other algebraic methods especially the method that Koohestani [20] developed for circulant structures. This method can be applied to other type of structures such as frames, space structures and finite element models.

2. FORMULATION OF THE FORCE METHOD BY SVD

Equilibrium equation of a structure in global coordinate system between external forces and internal stresses is given by:

$$H.\ r = p \tag{1}$$

where H is the n×m equilibrium matrix, r is the m×1 internal or element force vector, p is the m×1 external load vector. Equilibrium matrix in terms of displacement vector in local and global coordinate system can be expressed as:

$$H^T.\ \Delta = \delta \tag{2}$$

where H^T is the compatibility matrix which can easily be shown by a virtual work argument (Kaveh [21]), Δ and δ are nodal displacement vectors in global and local coordinate systems, respectively. Decomposition form of SVD for equilibrium matrix is given by:

$$H = U.\ W.\ V^T \tag{3}$$

where U and V contain the left and right orthogonal singular vectors whose ranks are n×n and m×m, respectively, W is a diagonal matrix containing singular values of equilibrium matrix and some rows or columns of that are zeros. Columns corresponding to zero eigenvalues in matrices U and V have specific properties shown in Fig. 1.

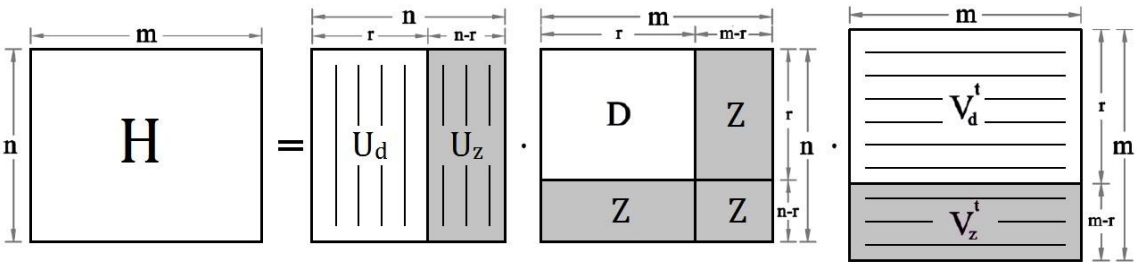

Figure 1. Schematic view of singular value decomposition of the equilibrium matrix of unstable structures

In Figure 1 the SVD form of the equilibrium matrix of an unstable structure is shown. In this decomposition, r = rank (W), Z is the matrix of zero values, D is the nonzero eigenvalues of matrix W, n is the number of degrees of freedom in global coordinate system, m is the number of internal forces of elements. U_z and V_z are representations of the loads which do not satisfy equilibrium conditions and the self-stress forces of the elements.

The SVD form of stable structures is as follows:

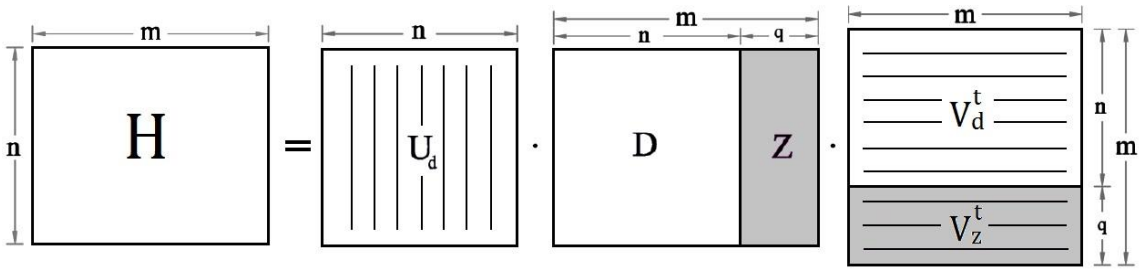

Figure 2. Schematic view of singular value decomposition of the equilibrium matrix of stable structures

One of the well-known applications of SVD is calculating pseudo inverse of a matrix. After decomposing the equilibrium matrix, its pseudo inverse is given by:

$$\text{pinv}(H) = V_d.\ D^{-1}.\ U_d^T \tag{4}$$

where pinv is the pseudo inverse of matrix H, D is a square matrix containing nonzero singular values of equilibrium matrix. Matrices U, W, V can be decomposed as follows:

$$U = [U_d \mid U_z] \quad , \quad W = [D \mid Z] \quad , \quad V = [V_d \mid V_z] \tag{5}$$

The internal forces of elements are calculated by:

$$r = \text{pinv}(H).\ p + V_z.\ \alpha \tag{6}$$

where α is the redundant vector, $V_z.\alpha$ is the representation of the self-stress forces and other parameters are defined before. These forces are written by applying compatibility conditions in nodes of structure defined by nodal compatibility equations. Compatibility equations in nodes of a structure could be as follows:

$$V_z^T.\ \delta = Z \tag{7}$$

In the local system, it is changed to:

$$\delta = F_m.\ r \tag{8}$$

where F_m is the flexibility matrix of the elements of the structure, δ is nodal displacement vector of the elements in local coordinate system. By substitution of the last three equations, following equation is obtained:

$$V_z^T.\ F_m.\ [\text{pinv}(H).\ p + V_z.\ \alpha] = Z \tag{9}$$

By simplifying, the following relation we have:

$$\alpha = - (V_z^T . F_m . V_z)^{-1} . [V_z^T . F_m . \text{pinv}(H) . p] \tag{10}$$

Matrices B_0 and B_1 of the structure obtained from the SVD form are given by:

$$B_0 = \text{pinv}(H), B_1 = V_z \tag{11}$$

3. BLOCK-DIAGONALIZING THE EQUILIBRIUM MATRIX

Equilibrium matrix of symmetric structures could be block-diagonalized by group representation theory. Kangwai and Guest [13] has described this method in details. Kronocker product can be used to block-diagonalize circulant structures. This paper will discuss how to block-diagonalize an equilibrium matrix by Kronocker product. First the Kronocker product for diagonalizing symmetric and block matrices is explained, then it is used to block-diagonalize equilibrium matrix of circulant structures. In matrix algebra, if eigenvectors of a square matrix make an identity orthogonal set, a square matrix can be made a diagonalized matrix by its eigenvalues. Hermitian matrices can be diagonalized while its diagonal contains the eigenvalues of the matrix.

If matrix M is considered as a Hermitian matrix, it can be diagonalized by pre- and post-multiplying with the matrix M (U^T.M.U). For calculating matrix U, the SVD method can be used. If the matrix M is assumed to be a block matrix, it can be written as follows by considering the Kronocker product:

$$M = A_1 \otimes B_1 \tag{12}$$

Matrix A_1 is a Hermitian matrix which can be diagonalized as D_{A1} and $D_{A1} \otimes B_1$ is the block-diagonalized form of matrix M. Matrix M is considered as summation of two Kronocker products is given by:

$$M = A_1 \otimes B_1 + A_2 \otimes B_2 \tag{13}$$

By assuming that matrix P diagonalizes ordinary matrices A_1 and A_2, it can be shown that matrix $U = P \otimes I$ block-diagonalizes matrix M. It can be proved that the matrix U^T.M.U is block-diagonalized.

$$(A \otimes B)^T = A^T \otimes B^T \text{ and } (A \otimes B)(C \otimes D) = A. C \otimes B. D \tag{14}$$

The proof is given by:

$$\begin{aligned} U^T.\ M.\ U &= (P^T \otimes I^T).\ (A_1 \otimes B_1 + A_2 \otimes B_2)(P \otimes I) \\ &= [(P^T.\ A_1) \otimes (I.\ B_1) + (P^T.\ A_2) \otimes (I.\ B_2)].\ (P \otimes I) \\ &= (P^T.\ A_1.\ P) \otimes (B_1.\ I) + (P^T.\ A_2.\ P) \otimes (B_2.\ I) \\ &= (P^T.\ A_1.\ P) \otimes (B_1) + (P^T.\ A_2.\ P) \otimes (B_2) \end{aligned} \tag{15}$$

Matrix P diagonalizing the matrices A_1 and A_2:

$$P^T.\ A_1.\ P = D_{A1},\ P^T.\ A_2.\ P = D_{A2} \tag{16}$$

Therefore

$$U^T.\ M.\ U = D_{A1} \otimes B_1 + D_{A2} \otimes B_2 \tag{17}$$

Consequently matrix M is block-diagonalized.

Now the first assumption which says that an orthogonal matrix U can be found to diagonalize either A_1 or A_2 is checked. The required condition for diagonalizing two matrices simultaneously is expressed in the following theorem.

Theorem: Necessary and sufficient condition for diagonalizing two Hermitian matrices A_1 and A_2 simultaneously by using an orthogonal matrix is given by:

$$A_1.\ A_2 = A_2.\ A_1 \tag{18}$$

For proof the reader may refer to Kaveh and Rahami [14].

If matrix A_1 is an identity matrix and matrix A_2 is a permutation matrix instead of being Hermitian matrix, the theorem is confirmed and their summation can be block-diagonalized. According to the above arguements, the required matrix U for block-diagonalizing matrix M can be obtained by using an identity orthogonal matrix made up of singular value decomposition of linear combination of two matrices A_1 and A_2. It will be illustrated how to block-diagonalize such matrices.

In the Kronocker product used for circulant structures, nodal ordering and then member ordering have to be in a manner which nodes and elements of each repeated block of structure ordered firstly. The resulting equilibrium matrix of this ordering is shown in Figure 3.

$$H = \begin{bmatrix} A_1 & & & & & B_n \\ B_1 & A_2 & & & & \\ & B_2 & A_3 & & & \\ & & B_3 & \cdots & & \\ & & & \cdots & A_{n-1} & \\ & & & & B_{n-1} & A_n \end{bmatrix}$$

Figure 3. The equilibrium matrix of a circulant structure using the discussed ordering

For having the same blocks in the above equilibrium matrix, the coordinate system should be rotated proportional to the angle of rotation. These changes are saved in a matrix called transformation matrix T. The equilibrium matrix multiplied by transformation matrix is shown in Figure 4.

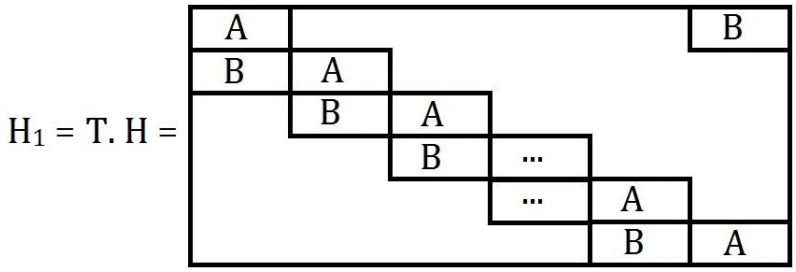

Figure 4. The equilibrium matrix of a circulant structure multiplied by the transformation matrix

Now, the equilibrium matrix can be decomposed as the sum of two Kronocker products as follows:

$$H_1 = I_1 \otimes A + I_2 \otimes B \tag{19}$$

where pattern of matrices I_1 and I_2 are depicted in Figure 5.

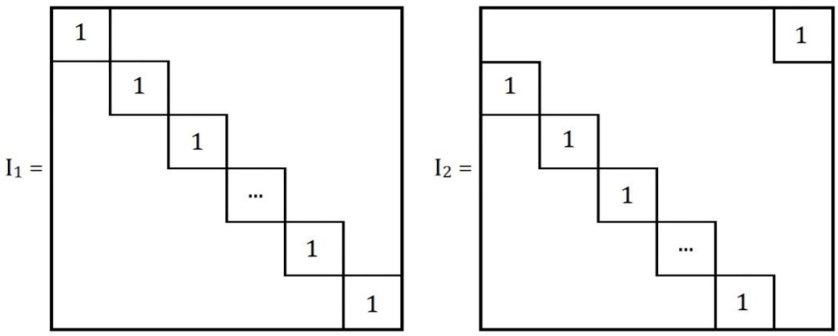

Figure 5. Patterns of the matrices I_1 and I_2 obtained by decomposing the equilibrium matrix

As pointed out before, there is an identity orthogonal matrix P to block-diagonalize the matrices I_1 and I_2. Matrix P is calculated by using singular value decomposition of linear combination of two matrices I_1 and I_2. Therefore it can be written as:

$$P^T.\ I_1.\ P = \text{Diagonalized matrix},\ P^T.\ I_2.\ P = \text{Diagonalized matrix} \tag{20}$$

Finally the equilibrium matrix can be block-diagonalized by using matrices U_1 and U_2 as:

$$U_1 = P \otimes I_n,\ U_2 = P \otimes I_m,\ U_1^T.\ H_1.\ U_2 = \text{Block-diagonalized matrix} \tag{21}$$

The Kronocker product described by Kaveh and Rahami [13], was applicable to square matrices while in this paper it can be applied to rectangular matrices as the equilibrium matrix.

4. FORMULATION OF THE FORCE METHOD BY A BLOCK-DIAGONALIZED EQUILIBRIUM MATRIX

For symmetric structures especially circulant ones, the equilibrium matrix can be block-diagonalized by concept of group representation theory and Kronocker product. The modified form of matrices in force method are given as:

$$\begin{aligned} \bar{H} &= U_1^T.\ T.\ H.\ U_2 \\ \bar{P} &= U_1^T.\ T.\ P \\ \bar{r} &= U_2^T.\ r \\ \bar{F} &= U_2^T.\ F.\ U_2 \end{aligned} \tag{22}$$

5. EXAMPLES AND DISCUSSION

After block-diagonalizing an equilibrium matrix, every algebraic method can be applied to each block. This reduces the dimensions of the matrix and acceleration in the analysis operations. The SVD method not only has a better accuracy results but also gives an orthogonal null basis matrix. This method has been compared previously to other algebraic methods by Koohestani [20], and the only disadvantage of that, has been high computational time. Block-diagonalizing the equilibrium matrix of the symmetric structures and using The SVD method on each block can improve the problem of high computational time. The following examples are provided to illustrate the efficiency of the present method in the circulant structures.

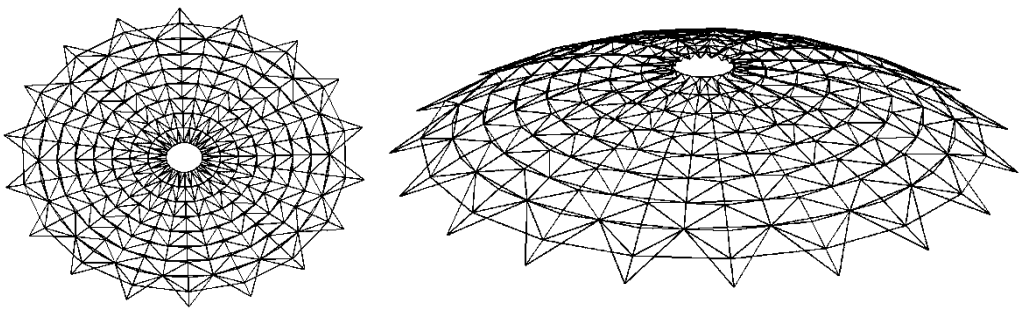

Figure 6. 1206-element Truss dome with 18 repeated substructures

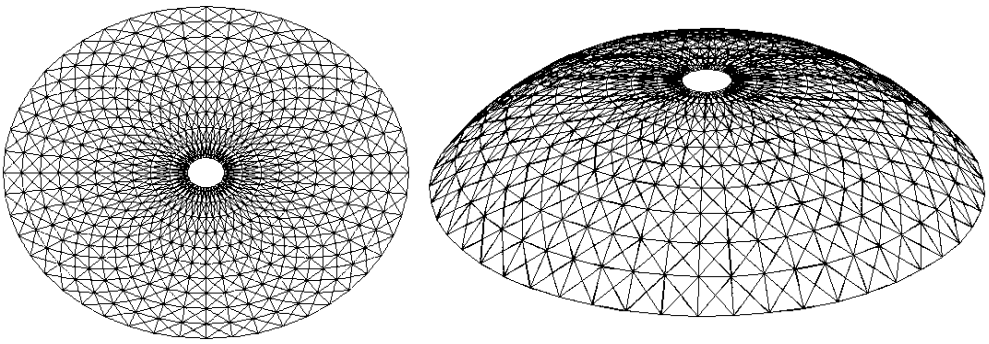

Figure 7. 1968-element Truss dome with 48 repeated substructures

It should be note that in order to perform a detailed structural analysis, the elements are assumed to be constructed of steel pipes with a cross sectional area of 2e-4 m^2 and E = 2e+8 kN/m^2 and other geometric properties are as given in Table 1. External load in the first example is 1000 kN in the direction of gravity and in second example is 1 kN in the main coordinate directions. The results are compared to the results of SAP 2000 software and for four elements are shown in Table 2. These element forces have the unit of kN and these are completely identical to the results of the SAP 2000 software.

Table 1: Specifications of two examples

	Example 1	Example 2
Total number of elements	1062	1968
Total number of nodes	324	528
Total number of restrained nodes	18	48
Degree of statical indeterminacy	144	528
Total number of elements in the main substructure	59	41
Number of identical structures	18	48
Angle between similar substructures (in degree)	20	7.5

The block-diagonalized flexibility matrices whose patterns are shown in Figures 8 and 9 illustrate the performance of this method. For checking the accuracy of this method, condition number (ratio of the biggest eigenvalue to the smallest one [15]) of flexibility matrices are calculated and then compared to another algebraic method. Condition numbers for the first and second examples by using SVD method are 4.9 and 22.9 respectively, while for LU decomposition method cannot be calculated. By increasing the number of rotations, the computation time of analysis is reduced shown in Figure 10.

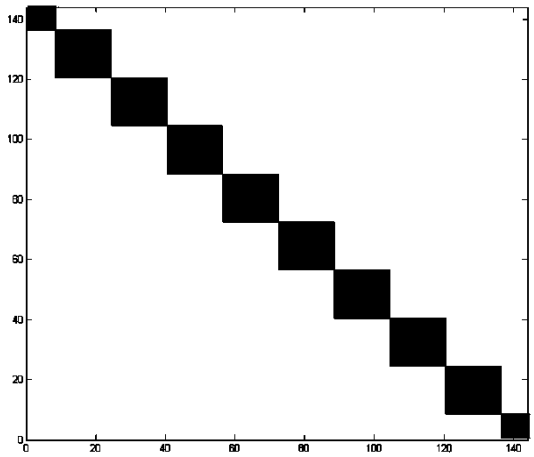

Figure 8. Pattern of the flexibility matrix for 1206-element Truss dome

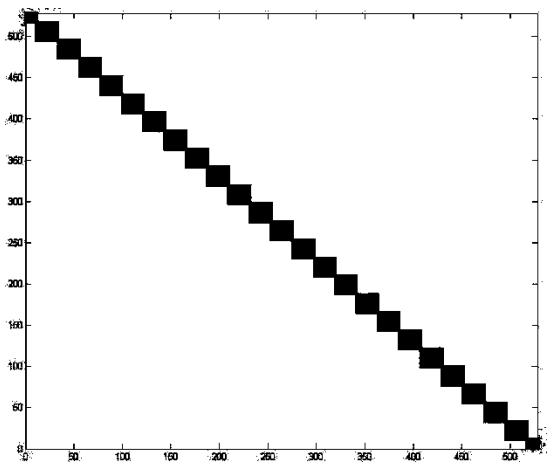

Figure 9. Pattern of the flexibility matrix for 1968-element Truss dome

Table 2: Element forces using the present method identical to those of the SAP 2000 software

	Example 1	Example 2
Element 1	11850.107	-0.848
Element 315	-141.768	-7.622
Element 785	-467.835	-8.307
Element 1042	8831.470	53.330

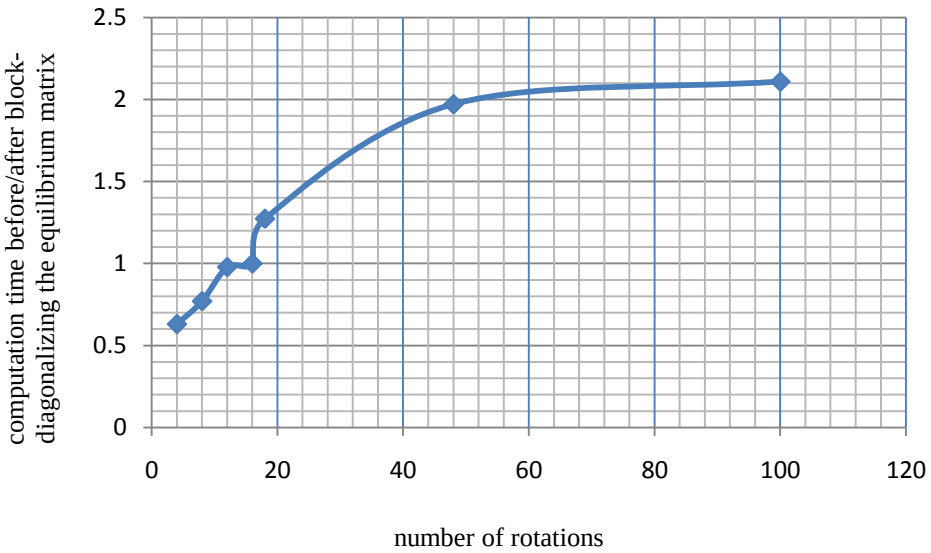

Figure 10. Comparison of computation time to the number of rotations

6. CONCLUSION

In this paper it is explained how to block-diagonalize the equilibrium matrix of circulant structures by using Kronocker product. After block-diagonalizing the equilibrium matrix, the singular value decomposition method is applied on each block to calculate the element forces by force method analysis. This method can be applied to other type of circulant structures such as finite element problems. The SVD method is almost an accurate method which disadvantage was being time consuming. This problem is solved by block-diagonalizing the equilibrium matrix. Finally the flexibility matrix of these structures is block-diagonalized and the number of nonzero entries is increased which can be applied to null basis problems.

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