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FREQUENCY ANALYSIS OF TRAPEZOIDAL PLATES AND MEMBRANE USING DISCRETE SINGULAR CONVOLUTION

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Abstract

In the present study, Discrete Singular Convolution (DSC) method is developed for free vibration analysis of plates and membranes with trapezoidal shape. The straight-sided quadrilateral domain is mapped into a square domain in the computational space using a four-node element. By using the geometric transformation, the governing equations and boundary conditions of the plate are transformed from the physical domain into a square computational domain. Numerical examples illustrating the accuracy and convergence of the DSC method for trapezoidal plates and membranes are presented. The results obtained by DSC method were compared with those obtained by the other numerical and analytical methods.

Keywords: Discrete singular convolution; free vibration; geometric mapping; trapezoidal plate; membrane

1. Introduction

Membranes and plates are widely used in various engineering applications. Membrane structures are frequently encountered in most practical acoustical and technological applications. Hence, many researchers in this area have been carried out. The analysis of straight-sided quadrilateral plates has been the subject of the research of structural and mechanical engineering. Cubic serendipity shape functions were first employed for arbitrary shaped general plates by finite strip method [1,2]. Following, Wang et al. [3] and Geannakakes [4] also used a similar approach to analyze irregular plates using the finite strip method in conjunction with orthogonal polynomials and linear serendipity shape functions, respectively. Liew and Han [4] introduced a mapping technique to apply the differential quadrature (DQ) method for analysis of. Blending functions was employed by Shu *et al*. [5] for vibration analysis of curvilinear quadrilateral plates using the DQ method. Bert and Malik [6] improved the numerical accuracy by using the DQ method for plate vibration with irregular domain. Following, a DQ solution for straight-sided quadrilateral plates has also been presented by Karami and Malekzadeh [7,8]. Detailed reviews on vibration analysis of

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plates have been made by Leissa [9-13]. The primary objective of this study is to give a numerical solution of free vibration analysis of trapezoidal plates and membranes. For this purpose, the straight-sided quadrilateral domain is mapped into a square domain in the computational space using a four-node element.

2. Discrete Singular Convolution

Discrete Singular Convolutions (DSC) algorithm introduced by Wei [14]. Wei and his co-workers first applied the DSC algorithm to solve some mechanics problem [15-18]. Zhao *et al.* [14,20] analyzed the high frequency vibration of plates and plate vibration under irregular internal support using DSC algorithm. Numerical solutions of free vibration problem of rotating and laminated conical shells and plates on elastic foundation have been proposed by the present author [21-25]. In a general definition, numerical solutions of differential equations are formulated by some singular kernels. A singular convolution can be defined by [14]

$$F(t) = (T * \eta)(t) = \int_{-\infty}^{\infty} T(t-x)\eta(x)dx \tag{1}$$

Where $T(t-x)$ is a singular kernel. For example, we have the singular kernels of delta type as [15]:

$$T(x) = \delta^{(n)}(x)\,;\ \ (\text{n} = 0, 1, 2, ...,). \tag{2}$$

Kernel $T(x) = \delta(x)$ is important for interpolation of surfaces and curves and $T(x) = \delta^{(n)}(x)$ for n>1 is essential for numerically solving differential equations. Recently, the use of some new kernels and regularizer such as delta regularizer [15-20] was proposed to solve applied mechanics problem. The Shannon's kernel is regularized as [12]

$$\delta_{\Delta,\sigma}(x-x_k) = \frac{\sin[(\pi/\Delta)(x-x_k)]}{(\pi/\Delta)(x-x_k)} \exp\left[-\frac{(x-x_k)^2}{2\sigma^2}\right];\ \sigma>0. \tag{3}$$

Where Δ is the grid spacing. Eq. (3) can also be used to provide discrete approximations to the singular convolution kernels of the delta type [18]

$$f^{(n)}(x) \approx \sum_{k=-M}^{M} \delta_{\Delta}(x-x_k) f(x_k)\ , \tag{4}$$

where $\delta_{\Delta}(x-x_k)=\Delta\delta_{\alpha}(x-x_k)$ and superscript (n) denotes the n*th*-order derivative, and $2M+1$ is the computational bandwidth which is centred around x and is usually smaller than the whole computational domain. In the DSC method, the function $f(x)$ and its derivatives with respect to the x coordinate at a grid point x_i are approximated by a linear sum of discrete values $f(x_k)$ in a narrow bandwidth [$x-x_M$, $x+x_M$]. This can be expressed as

$$\left.\frac{d^n f(x)}{d x^n}\right|_{x=x_i} = f^{(n)}(x) \approx \sum_{k=-M}^{M} \delta_{\Delta,\sigma}^{(n)}(x_i - x_k) f(x_k)\,;\ (n=0,1,2,...,). \tag{5}$$

Where superscript n denotes the nth-order derivative with respect to x. The x_k is a set of discrete sampling points centred around the point x, σ is a regularization parameter, Δ is the grid spacing, and $2M+1$ is the computational bandwidth, which is usually smaller than the size of the computational domain.

3. Geometric Mapping for Straight-Sided Plates

Consider an arbitrary straight-sided quadrilateral plate in the Cartesian x-y plane, as shown in Figure 1(a). The geometry of this plate can be mapped into a rectangular plate in the natural ξ-η plane, as shown in Figure 1(b). By employing the following transformation equations the physical domain is mapped into the computational domain

$$x = \sum_{i=1}^{N} x_i \Phi_i(\xi,\eta) \quad \text{and} \quad y = \sum_{i=1}^{N} y_i \Phi_i(\xi,\eta) \tag{6,7}$$

Where x_i and y_i are the coordinates of node i in the physical domain, N is the number of grid points, and $\Phi_i(\xi,\eta)$; i= 1,2,3,…,N are the interpolation or shape functions. These are given for node i [4];

$$\Phi_i(\xi,\eta) = \frac{1}{4}(1+\xi\,\xi_i)(1+\eta\,\eta_i) \tag{8}$$

Using the chain rule, the first-order, and second order derivatives of a function are given

$$\begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = [J_{11}]^{-1} \begin{Bmatrix} u_\xi \\ u_\eta \end{Bmatrix} \tag{9}$$

$$\begin{Bmatrix} u_{xx} \\ u_{yy} \\ 2u_{yx} \end{Bmatrix} = [J_{22}]^{-1} \begin{Bmatrix} u_{\xi\xi} \\ u_{\eta\eta} \\ 2u_{\xi\eta} \end{Bmatrix} - [J_{22}]^{-1}[J_{21}][J_{11}]^{-1} \begin{Bmatrix} u_{\xi} \\ u_{\eta} \end{Bmatrix} \tag{10}$$

where ξ_i and η_i are the coordinates of Node i in the ξ-η plane, and J_{ij} are the elements of the Jacobian matrix. The above transformations will be used later to transform the governing differential equations and related boundary conditions from the physical domain x-y into the computational domain ξ-η. Thus an arbitrary-shaped quadrilateral plate may be represented by the mapping of a square plate defined in terms of its natural coordinates.

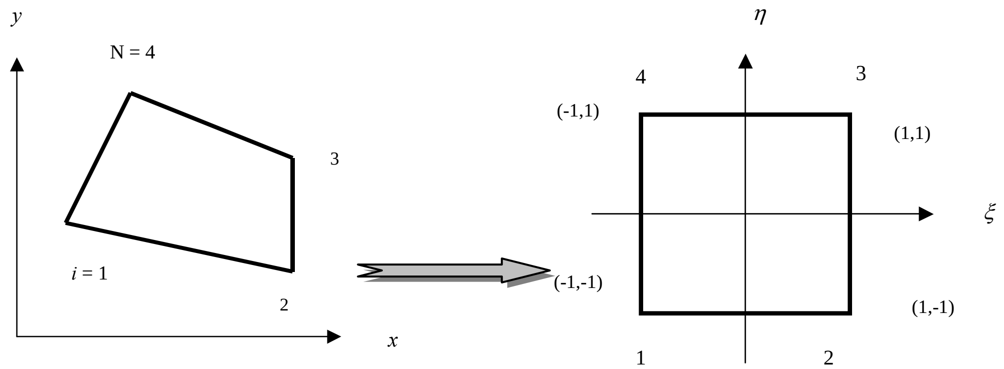

Figure 1. Mapping of arbitrary quadrilateral plates into natural coordinates

4. Fundamental Equation of Motion

4.1 Plate

The normalized governing differential equations for vibration of thin plates are given as

$$\frac{\partial^4 W}{\partial x^4} + 2\frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} = \Omega^2 W \tag{11}$$

Where $\Omega^2 = \rho h \omega^2 / D$. Also, D is the coefficient of the bending rigidity for plate, h is the plate thickness, N_x and N_y are the applied compressive loads in the respective x and y directions, q is the pressure, w is the deflection, ρ is the density, x and y are the midplane Cartesian coordinate.

$$\nabla^2(\bullet) = \frac{\partial^2(\bullet)}{\partial x^2} + \frac{\partial^2(\bullet)}{\partial y^2} \tag{12}$$

Where ∇^2 is the Laplace operator. Thus, Eq. (11) takes the following simple form:

$$\nabla^2\nabla^2(W_{XY}) = \Omega^2 W \tag{13}$$

Consider the following differential operators before discretizing the governing differential equations

$$\Re = \frac{\partial^2 W}{\partial x^2} \text{ and } S = \frac{\partial^2 W}{\partial y^2} \tag{14}$$

Thus, the fourth-order derivatives can be given in terms of the second order derivatives, that is,

$$\frac{\partial^4 W}{\partial x^4} = \frac{\partial^2}{\partial x^2}\Re \text{ and } \frac{\partial^4 W}{\partial y^4} = \frac{\partial^2}{\partial y^2} S \tag{15,16}$$

$$\frac{\partial^4 W}{\partial x^2 \partial y^2} = \frac{\partial^2}{\partial x^2}\left[\frac{\partial^2 w}{\partial y^2}\right] = \frac{\partial^2}{\partial x^2} S \tag{17}$$

After the transformation process, the following form can be given for the first-, second, and the fourth-order derivatives, respectively

$$\frac{\partial W}{\partial x} = [J_{11}]^{-1}\frac{\partial W}{\partial \xi} \text{ and } \frac{\partial W}{\partial y} = [J_{11}]^{-1}\frac{\partial W}{\partial \eta} \tag{18,19}$$

$$\frac{\partial^2 W}{\partial x^2} = [J_{22}]^{-1}\frac{\partial^2 W}{\partial \xi^2} - [J_{22}]^{-1}[J_{21}][J_{11}]^{-1}\frac{\partial W}{\partial \xi} \tag{20}$$

$$\frac{\partial^2 W}{\partial y^2} = [J_{22}]^{-1}\frac{\partial^2 W}{\partial \eta^2} - [J_{22}]^{-1}[J_{21}][J_{11}]^{-1}\frac{\partial W}{\partial \eta} \tag{21}$$

$$\frac{\partial^4 W}{\partial x^4} = \frac{\partial^2 \Re}{\partial \xi^2} = [J_{22}]^{-1}\frac{\partial^2 \Re}{\partial \xi^2} - [J_{22}]^{-1}[J_{21}][J_{11}]^{-1}\frac{\partial \Re}{\partial \xi} \tag{22}$$

$$\frac{\partial^4 W}{\partial y^4} = \frac{\partial^2 S}{\partial \eta^2} = [J_{22}]^{-1}\frac{\partial^2 S}{\partial \eta^2} - [J_{22}]^{-1}[J_{21}][J_{11}]^{-1}\frac{\partial S}{\partial \eta} \tag{23}$$

$$\frac{\partial^4 W}{\partial x^2 \partial y^2} = \frac{\partial^2 S}{\partial x^2} = [J_{22}]^{-1} \frac{\partial^2 S}{\partial \xi^2} - [J_{22}]^{-1}[J_{21}][J_{11}]^{-1} \frac{\partial S}{\partial \xi} \tag{24}$$

Using the differential operators in Eqs. (18-24), the normalized governing equation, i.e., Eq. (13), takes the following form

$$\frac{\partial^2 \Re}{\partial x^2} + 2\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} = \Omega^2 W \tag{25}$$

Employing the transformation and DSC rule, the governing Eq. (25) becomes

$$\begin{aligned} &[J_{22}]^{-1}\left[\sum_{k=-M}^{M} \delta^{(2)}_{\Delta,\sigma}(k\Delta\xi)\,\Re_{kj} + 2\sum_{k=-M}^{M} \delta^{(2)}_{\Delta,\sigma}(k\Delta\eta)\,\Re_{ik} + \sum_{k=-M}^{M} \delta^{(2)}_{\Delta,\sigma}(k\Delta\eta)\,S_{ik}\right] \\ &-[J_{22}]^{-1}[J_{21}][J_{11}]^{-1}\left(\sum_{k=-M}^{M} \delta^{(1)}_{\Delta,\sigma}(k\Delta\xi)\,\Re_{kj} + 2\sum_{k=-M}^{M} \delta^{(1)}_{\Delta,\sigma}(k\Delta\eta)\,\Re_{ik} \right. \\ &\left. + \sum_{k=-M}^{M} \delta^{(2)}_{\Delta,\sigma}(k\Delta\eta)\,S_{ik}\right) = \Omega^2 W_{ij} \end{aligned} \tag{26}$$

For convenience and simplicity, the following new variable is introduced:

$$\Im = (k\Delta\xi)\,\Re_{kj} + 2(k\Delta\xi)\,\Re_{ik} + (k\Delta\eta)\,S_{ik} \tag{27}$$

Such that the governing equations of plate for free vibration can be expressed as

$$[J_{22}]^{-1}\left[\sum_{k=-M}^{M} \delta^{(2)}_{\Delta,\sigma}\Im\right] - [J_{22}]^{-1}[J_{21}][J_{11}]^{-1}\left[\sum_{k=-M}^{M} \delta^{(1)}_{\Delta,\sigma}\Im\right] = \Omega^2 W_{ij} \tag{28}$$

In order to the discretized form of Eq. (13) in its natural coordinate, we apply Eqs. (28) to below equation

$$\nabla^4 (W_{\xi\eta}) = \nabla^2\nabla^2 (W_{\xi\eta}) = \Omega^2 W \tag{29}$$

On substituting Eq. (28) into Eq. (29) the governing equation can now be given by

$$\begin{aligned} &\left([J_{22}]^{-1}\left[\sum_{k=-M}^{M} \delta^{(2)}_{\Delta,\sigma}\Im\right] - [J_{22}]^{-1}[J_{21}][J_{11}]^{-1}\left[\sum_{k=-M}^{M} \delta^{(1)}_{\Delta,\sigma}\Im\right]\right. \\ &\left.\times [J_{22}]^{-1}\left[\sum_{k=-M}^{M} \delta^{(2)}_{\Delta,\sigma}\Im\right] - [J_{22}]^{-1}[J_{21}][J_{11}]^{-1}\left[\sum_{k=-M}^{M} \delta^{(1)}_{\Delta,\sigma}\Im\right]\right) = \Omega^2 W_{ij} \end{aligned} \tag{30}$$

Therefore, the governing equation is given by the matrix notation as

$$(\mathbf{D}_{\xi}^{4} \otimes \mathbf{I}_{\eta} + 2\mathbf{D}_{\xi}^{2} \otimes \mathbf{D}_{\eta}^{2} + \mathbf{I}_{\xi} \otimes \mathbf{D}_{\eta}^{4})\mathbf{W} = \Omega^{2}\mathbf{W}. \tag{31}$$

Where $\mathbf{I}_{\xi}$ and $\mathbf{I}_{\eta}$ are the $(N_r + 1)^2; (r = \xi, \eta)$ unit matrix and $\otimes$ denotes the tensorial product. Two types of boundary conditions, i.e., simply supported (S) and clamped (C) are taken into consideration. Following, the related formulations and their DSC form are given in detail.

i) For simply supported edge (S)

$$W = 0\,,\ -D(\frac{\partial^{2}W}{\partial n^{2}} + v\frac{\partial^{2}W}{\partial s^{2}}) = 0. \tag{32}$$

ii) For clamped edge (C)

$$W = 0\,,\ \frac{\partial W}{\partial n} = 0. \tag{33}$$

Where *n* and *s* denote the normal and tangential directions of the plate, respectively. It is known that, to obtain a unique solution for a differential equation, appropriate boundary conditions must be satisfied. In applying the DSC method Wei et al. [17,18] and Zhao *et al*. [19-21] proposed a practical method in applying the simply supported and clamped boundary conditions. We used the same procedure proposed by Wei et al. [17] and Zhao *et al*. [20], in this study. Finally, after boundary conditions being implemented, the differentiation matrix (for example vibration case), in Eq. (31) is given as $D^{*n}_{r}(r = X, Y; n = 1,2,...)$. Here $\mathbf{D}^{*n}_{r}$ is a $(N-2)\times(N-2)$ differential matrix and superscript * is introduced to avoid confusion in differential matrix with D_{r}^{n} in Eq. (31). Thus, Eq. (31) is rewritten as

$$(\mathbf{D}^{*4}_{\xi} \otimes \mathbf{I}\eta + 2\mathbf{D}^{*2}_{\xi} \otimes \mathbf{D}^{*2}_{\eta} + \mathbf{I}_{\xi} \otimes \mathbf{D}^{*4}_{\eta})W = \Omega^{2}W. \tag{34}$$

in which *W* is the column vector, that is,

$$\mathbf{W} = (W_{1,1},...W_{1,N-2}, W_{2,1},...W_{N-2,N-2})^{T} \tag{35}$$

4.2 Membrane

Membranes are widely used in various engineering applications such as the design stage of microphones, pumps, pressure regulators, and other acoustical applications [21-31]. The governing differential equation for free vibration of membranes is [26]

$$\frac{\partial^2 W}{\partial X^2} + \lambda^2 \frac{\partial^2 W}{\partial Y^2} + \Omega^2 W = 0, \tag{36}$$

where W is the transverse deflection, ρ is the mass per unit area, ω is the circular frequency, and T is the tension per unit length. The density of the membrane is the linear function of the x. In Eq. (36) the non-dimensional variables have been used given below

$$X = x/a, Y = y/b, \ \Omega^2 = \rho\omega^2 a^2 / T, \lambda = a/b \tag{37}$$

Applying the discrete singular convolution to the governing equation yields

$$\sum_{k=-M}^{M} \delta_{\Delta,\sigma}^{(2)}(k\Delta x) W_{i+k,j} + \lambda^2 \sum_{k=-M}^{M} \delta_{\Delta,\sigma}^{(2)}(k\Delta y) W_{i,j+k} + \Omega^2 W_{ij} = 0, \tag{38}$$

The boundary conditions are as follows:

$$W=0 \text{ at edges} \tag{39}$$

Employing the transformation rule, the governing Eq. (38) becomes,

$$[J_{22}]^{-1} \sum_{i=-M}^{M} \delta^{(2)}{}_{\Delta,\sigma}(k\Delta\xi) W_{ik}$$

$$- [J_{22}]^{-1}[J_{21}][J_{11}]^{-1} \sum_{i=-M}^{M} \delta^{(1)}{}_{\Delta,\sigma}(k\Delta\xi) W_{ik} + \lambda^2 \left[[J_{22}]^{-1} \sum_{i=-M}^{M} \delta^{(2)}{}_{\Delta,\sigma}(k\Delta\eta) W_{jk} \right]$$
$$- \lambda^2 \left[[[J_{22}]^{-1}[J_{21}][J_{11}]^{-1} \sum_{i=-M}^{M} \delta^{(1)}{}_{\Delta,\sigma}(k\Delta\eta) W_{jk} \right] + \Omega^2 W_{ij} = 0 \tag{40}$$

5. Numerical Results

The results given in this section are aimed to illustrate the numerical accuracy of the proposed DSC based coordinate transformation method. The plates of various geometries are designated by the boundary conditions at their edges (Figures 2-3). For example, the symbol CSCS trapezoidal plate indicates that the trapezoidal plate would have the parallel edges clamped (C) and the other two nonparallel edges simply supported (S). The results are listed in Table 1 are for trapezoidal plate of Figure 2 and having three different boundary conditions. The following geometric properties are used for the trapezoidal plate: 2h/c=1.5, d/c=0.4. The results are compared with those obtained by Bert and Malik [6]. Natural frequency parameters of trapezoidal plate are given in Table 2, it is observed that a good agreement between the present calculated results and the results of literature [15] has been obtained. It can be observed that the rate convergence of DSC technique is excellent and comparison agrees very well. Non-dimensional frequencies of symmetric SSSS trapezoidal plate for different geometric parameter are given in Table 2. In general, the frequencies increase with the increasing of a/d ratios.

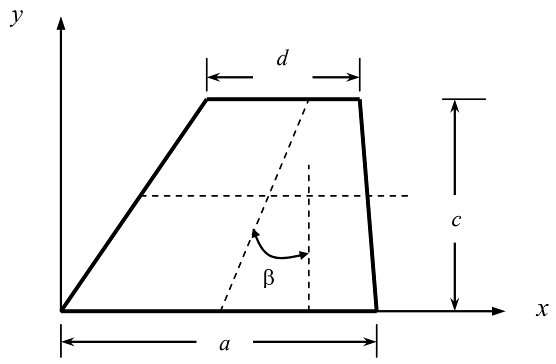

Figure 2. An unsymmetrical trapezoidal plate

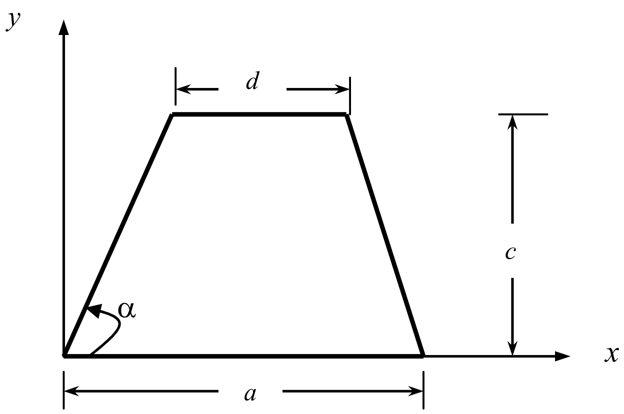

Figure 3. Symmetric trapezoidal plate

The fundamental frequency values are listed in Table 3 for different values of *c/a* and α for simply supported trapezoidal plate (Figure 3). In general, the values of frequency increase with an increase in the *c/a* ratio for plates with different value of α. This increasing of the frequency is more significant for α than the *c/a* ratio.

Free vibration analysis of trapezoidal membrane (Figure 4) is considered. The results obtained by the present method are compared with the finite element solution [26]. The frequency values are given in Table 4. The results are matching very well with the results given by Kang and Lee [26].

Table 1. Natural frequencies ($\Omega = \omega a^2 / \pi^2 \sqrt{\rho h / D}$) of trapezoidal plate (2h/c=3.0;c/d=2.5; β=0)

Boundary conditions	**DQ results (N_ζ=N_η= 17) Bert and Malik [6]**			**Present DSC results N_ζ=N_η= 16**		
	Mode sequence			**Mode sequence**		
	1	**2**	**3**	**1**	**2**	**3**
SSSSS	5.388	9.421	14.676	5.389	9.422	14.680
CCCC	10.427	15.563	21.476	10.427	15.565	21.478
SCSC	9.443	14.386	19.897	9.449	14.401	19.906

Table 2. Non-dimensional frequencies of symmetric SSSS trapezoidal plate

a/d	**a/c**			
	0.5	1.0	1.5	2.0
1	7.032	4.450	3.771	3.514
1.5	8.103	4.962	4.055	3.849
2.0	8.694	5.288	4.268	3.837
2.5	9.045	5.548	4.451	3.968

Table 3. Non-dimensional fundamental frequencies ($\Omega = \omega a^2 \sqrt{\rho / D}$) of symmetric SSSS trapezoidal plate

c/a	**α**			
	60	65	70	80
0.125	3.3515	2.1849	1.3302	0.3125
0.25	3.5882	2.3251	1.4094	0.3283
0.5	5.1403	3.1385	1.8216	0.4008

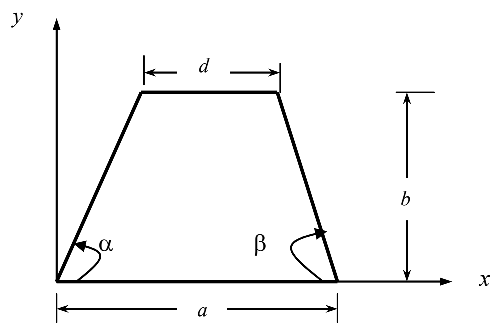

Figure 4. Trapezoidal membrane

Table 4. Comparison of frequency values of the trapezoidal membrane (a/b=2.0; β= 70°; α=60°)

Mode	Methods			
	Ref. 33	Present DSC N=11	Present DSC N=13	Present DSC N=15
1	3.81	3.83	3.82	3.82
2	5.29	5.30	5.27	5.27
3	6.58	6.58	6.56	6.56
4	7.07	7.07	7.05	7.05

6. Concluding Remarks

In the present study, using the DSC method, a numerical approach for the free vibration analysis of trapezoidal plates and membrane is presented. By using the geometric transformation, the governing equations and boundary conditions of the plate are transformed from the physical domain into a square computational domain. Several examples were worked to demonstrate the convergence of the method. Excellent convergence behavior and accuracy in comparison with exact results or results obtained by other numerical methods were obtained.

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References

1. Li WY, Cheung YK, Tham LG. Spline finite strip analysis of general plates, *Journal of Engineering Mechanics, ASCE*, No. 1, **112**(1968)43-54.
2. Cheung YK, Tham LG, Li WY. Free vibration and static analysis of general plates by

spline finite strip method, *Computatinal Mechanics*, **3**(1988)187-97.

3. Wang G, Cheng-Tzu, Hsu T. Static and dynamic analysis of arbitrary quadrilateral flexural plates by B_3-spline functions, *International Journal of Solids and Structures*, **31**(1994)657-67.
4. Liew KM, Han JB. A four–note differential quadrature method for straight-sided quadrilateral Reissner/Mindlin plates, *Communications in Numerical Methods in Engineering*, **13**(1997)73-81.
5. Shu C, Chen W, Du H. Free vibration analysis of curvilinear quadrilateral plates by the differential quadrature method, *Journal of Computational Physics*, **163**(200)452-66.
6. Bert CW, Malik M. The differential quadrature method for irreqular domains and application to plate vibration, *International Journal of Mechanical Sciences*, No. 6, **38**(1996)589-606.
7. Karami G, Malekzadeh P. Static and stability analyses of arbitrary straight-sided quadrilateral thin plates by DQM, *International Journal of Solids and Structures*, **39**(2002)4927-47.
8. Karami G, Malekzadeh P. An efficient differential quadrature methodology for free vibration analysis of arbitrary straight-sided quadrilateral thin plates, *Journal of Sound and Vibration*, **263**(2003)415-42.
9. Wei GW. A new algorithm for solving some mechanical problems, *Journal of Computer Methods in Applied Mechanics and Engineering*, **190**(2001)2017-30.
10. Wei GW. Vibration analysis by discrete singular convolution*, Journal of Sound and Vibration* **244**(2001)535-53.
11. Wei GW. Discrete singular convolution for beam analysis, *Engineering Structures,* **23**(2001)1045-53.
12. Wei GW, Zhao YB, Xiang Y. Discrete singular convolution and its application to the analysis of plates with internal supports. Part 1: Theory and algorithm, *International Journal of Numerical Methods in Engineering*, **55**(2002)913-46.
13. Wei GW, Zhao YB, Xiang Y. A novel approach for the analysis of high-frequency vibrations, *Journal of Sound and Vibration*, No. 2, **257**(2002)207-46.
14. Zhao YB, Wei GW, Xiang Y. Discrete singular convolution for the prediction of high frequency vibration of plates, *International Journal of Solids Structures*, **39**(2002)65-88.
15. Zhao YB, Wei GW, Xiang Y. Plate vibration under irregular internal supports *International Journal of Solids Structures,* **39**(2002)1361-83.
16. Zhao YB, Wei GW. DSC analysis of rectangular plates with non-uniform boundary conditions, *Journal of Sound and Vibration,* No. 2, **255**(2002)203-28.
17. Civalek Ö. An efficient method for free vibration analysis of rotating truncated conical shells, *International Journal of Pressure Vessels and Piping,* **83**(2006)1-12.
18. Civalek Ö. The determination of frequencies of laminated conical shells via the discrete singular convolution method, *Journal of Mechanics of Materials and Structures*, No. 1, **1**(2006)165-92.
19. Civalek Ö. Nonlinear analysis of thin rectangular plates on Winkler-Pasternak elastic foundations by DSC-HDQ methods, *Applied Mathematical Modeling*, **31**(2007)606-24.
20. Civalek Ö. Free vibration analysis of composite conical shells using the discrete singular convolution algorithm, *Steel and Composite Structures*, No. 4, **6**(2006)353-66.

21. Civalek Ö. Three-dimensional vibration, buckling and bending analyses of thick rectangular plates based on discrete singular convolution method, *International Journal of Mechanical Sciences*, **49**(2007)752-65.
22. Civalek Ö. Numerical analysis of free vibrations of laminated composite conical and cylindrical shells: discrete singular convolution (DSC) approach, *Journal of Computational and Applied Mathematics*, **205**(2007)251-71.
23. Leissa AW. Vibration of plates, NASA, SP-160, 1969.
24. Leissa AW. Recent research in plate vibrations: classical theory, *The Shock and Vibration Digest*, No. 10, **9**(1977)13-24.
25. Leissa AW. Recent research in plate vibrations: complicating effects, *The Shock and Vibration Digest*, No. 11, **9**(1977)21-35.
26. Leissa AW. Plate vibration research, 1976-1980:classical theory, *The Shock and Vibration Digest*, No. 9, **13**(1981)11-12.
27. Leissa AW. Recent research in plate vibrations, 1981-1985: Part I. classical theory, *The Shock and Vibration Digest*, No. 2, **19**(1987)11-18.
28. Buchanan GR, Peddieson Jr J. Vibration of circular and annular membranes with variable density, *Journal of Sound and Vibration*, No. 2, **226**(1999)379-82.
29. Buchanan GR, Peddieson Jr J. A finite element in elliptic coordinates with application of membrane vibration, *Thin-Walled Structures*, **43**(2005)1444-54.
30. Buchanan GR. Vibration of circular membranes with linearly varying density along a diameter, *Journal of Sound and Vibration*, **280**(2005)407-14.
31. Willatzen M. Exact power series solutions for axisymmetric vibrations of circular and annular membranes with continuously varying density in the general case, *Journal of Sound and Vibration*, No. 5, **258**(2002)981-86.
32. Kang SW, Lee JM, Kang YJ. Vibration analysis of arbitrarily shaped membranes using non-dimensional dynamic influence function, *Journal of Sound and Vibration*, **221**(1999)117-32.
33. Kang SW, Lee JM. Application of free vibration analysis of membranes using non-dimensional dynamic influence function, *Journal of Sound and Vibration*, **234**(2000)455-70.
34. Wu WX, Shu C, Wang CM. Vibration analysis of arbitrarily shaped membranes using local radial basis function-based differential quadrature method, *Journal of Sound and Vibration*, **306**(2007)252-70.
35. Leung AYT, Zhu B, Zheng J, Yang H. A trapezoidal Fourier p-element for membrane vibrations, *Thin-Walled Structures*, **41**(2003)479-91.
36. Houmat A. A sector Fourier p-element for free vibration analysis of sectorial membranes, *Computers and Structures*, **79**(2001)1147-52.
37. Houmat A. Free vibration analysis of arbitrarily shaped membranes using the trigonometric p-version of the finite element method, *Thin-Walled Structures*, **44**(2006)943-51.
38. Masad JA. Free vibrations of a non-homogeneous rectangular membrane, *Journal of Sound and Vibration*, **195**(1996)674-78.