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ISOGEOMETRICAL SOLUTION OF LAPLACE EQUATION

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ABSTRACT

A solution for Laplace partial differential equation by using Spline basis functions is presented. The formulation is derived and its differences with the finite element method are explained. The effect of some of parameters such as the knot vector and grid of control points on the solution is investigated. Finally, a few examples are presented to demonstrate the efficiency of the method.

Keywords: Spline basis functions; Isogeometrical analysis; Laplace equation; Finite elements

1. INTRODUCTION

Most of problems faced in different disciplines of science and engineering are engaged with solving differential equations. Since only a very limited of these equations can be solved analytically, several numerical methods have been developed in the last few decades. Amongst the most popular of these methods the finite difference, finite element and the wide range of so called mesh-free methods can be mentioned. One of the drawbacks of all of these methods, less or more, is that some approximation is involved in the geometrical definition of the boundaries of the problem domain. Furthermore, the imposition of the essential boundary conditions on the boundaries cannot be accomplished exactly, especially in the mesh free methods. Another problem is adaptivity and refinement of the solution where in the finite element method requires several communications between the discretized geometry and the analysis tool which is quite costly [1].

To overcome these problems, and inspired by the developments in geometrical modeling and CAD (Computer Aided Design) description of complicated shapes, the idea of isogeometric analysis, following the research on using splines in finite element analysis [2], has recently been proposed by Hughes *et al* [3,9-13]. Due to some interesting properties of splines and NURBS (Non-Uniform Rational B-Splines) beside accurate definition of geometry, their basis functions can be employed in place of interpolation and approximation functions of finite elements and meshfree methods [20]. Especially when adaptive solution is

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intended, there are several ways to increase the accuracy of the solution with a complete control over the geometry, e.g. degree elevation of the basis functions, increasing the size of knot vectors by knot insertion and knot refinement, increasing the number of control points and modification of their position, or a combination of them which needs further research [4]. This is similar to the *h* and *p* adaptivity in the finite elements.

In this paper, based on the concept of isogeometrical analysis, an algorithm is developed for solving the Laplace equation with its application in the heat conduction problem, to study its performance. In this case, the solution might be imagined as a surface which can be generated by using Splines and NURBS. The *x* and *y* coordinates of the control points are assumed ***a priori*** and the z coordinates are calculated by using one of the conventional weighted residual or variational methods.

In Section 2, the main concepts of surface definition by Splines are briefly explained. Section 3 is devoted to the derivation of the formulation and the system of equations. In Section 4 the effect of different parameters on the solution of a typical example is investigated. Finally, Conclusions and proposed further research is the subject of Section 5.

2. SURFACE DEFINITON BY SPLINES

The formulation of Splines and NURBS can be found in several references such as [5,14, 15,18,19] and is briefly pointed here. Defining a B-spline surface, in its general form, requires the following [5]:

- A set of $(n_1+1)\times(n_2+1)$ control points $P_{i,j}, i=0,1,\dots,n_1;\ j=0,1,\dots,n_2$
- Two knot vectors U and V for each direction with m_1 and m_2 components, respectively, where $m_1=n_1+p+1$ and $m_2=n_2+q+1$.
- Basis functions of degrees p and q in horizontal and vertical directions.

The B-Spline surface is parametrically constructed as follows:

$$S(u,v)=\sum_{i=0}^{n1}\sum_{j=0}^{n2}N_{i,p}(u)N_{j,q}(v)P_{i,j}, \tag{1}$$

where *u* and *v* are the parameters.

2.1 Knot Vectors

The knot vectors are defined as $\mathrm{U}=\{u_0,u_1,\dots,u_{m_1}\}$ and $\mathrm{V}=\{v_0,v_1,\dots,v_{m_2}\}$ where u_i and v_j are a non-decreasing sequence of real numbers; i.e., $u_i\le u_{i+1}$, $i=0,1,2,\dots,m_1$ and $v_j\le v_{j+1}$,$j=0,1,2,\dots,m_2$. The u_i and v_i are called ***knots***, and U and V are the ***knot vectors***. When, for instance, for every *i* we have $u_{i+1}-u_i=u_i-u_{i-1}$ then U is a uniform knot vector and non-uniform vice versa.

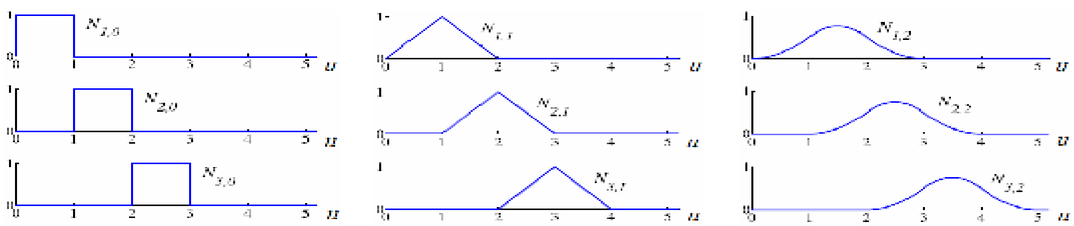
2.2 Basis Functions

The i-th B-Spline basis function of degree p (order $p+1$), denoted by $N_{i,p}(u)$, is defined recursively as:

$$N_{i,0}(u) = \begin{cases} 1 & \text{if } u_i \le u < u_{i+1} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$N_{i,p}(u) = \frac{u - u_i}{u_{i+p} - u_i} N_{i,p-1}(u) + \frac{u_{i+p+1} - u}{u_{i+p+1} - u_{i+1}} N_{i+1,p-1}(u) \tag{3}$$

For example, for $p = 0,1,2$ and using a uniform knot vector $U = \{0,1,2,3,4,5,\ldots\}$ the basis functions are shown in the Figure 1. For more details Reference [5] can be consulted.



(a) (b) (c)

Figure 1. Basis Functions of (a): orders 0, (b): order 1 and (c): order 2

3. DERIVATION OF NUMERICAL FORMULATION

Let's consider the following single valued partial differential equation to solve

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial u}{\partial y}\right) + pu + q = 0. \tag{4}$$

Following a conventional weak formulation procedure, the bilinear and linear part of the weak form can be constructed as

$$B(u,w) = \int_{\Omega_e} \left[k_x \frac{\partial w}{\partial x}\frac{\partial u}{\partial x} + k_y \frac{\partial w}{\partial y}\frac{\partial u}{\partial y} + puw\right] dxdy, \tag{5}$$

and

$$l(w) = \int_{\Omega_e} qw\,dxdy + \oint_{\Gamma_e} w\alpha_n\, d\Gamma \tag{6}$$

respectively, where w is the weight function and α_n is

$$\alpha_n = n_x\left(k_x \frac{\partial u}{\partial x}\right) + n_y\left(k_y \frac{\partial u}{\partial y}\right). \tag{7}$$

choosing w as a variation of u and having symmetry in B a functional Π can be constructed as

$$\Pi = \frac{1}{2}B(u,u) - l(u) \tag{8}$$

which its stationary condition is equivalent to solving the Equation (4). Substituting from (5) and (6) into (8) it follows

$$\Pi = \frac{1}{2}\int_{\Omega_e}\left[\left(\frac{\partial u}{\partial x}\right)^2 + k_y\left(\frac{\partial u}{\partial x}\right)^2 + pu^2\right]dxdy - \int_{\Omega_e} qudxdy - \oint_{\Gamma_e} u\alpha_n d\Gamma_e B(u,w) \tag{9}$$

Now, following a procedure analogous to the isoperimetric finite elements or meshfree methods, the geometrical variables, as well as the unknown function, are approximated by using the spline basis function as below

$$x(r,s) = \sum_{i=0}^{n}\sum_{j=0}^{m} N(r)_{i,p}N(s)_{j,q}X_{i,j} \tag{10}$$

$$y(r,s) = \sum_{i=0}^{n}\sum_{j=0}^{m} N(r)_{i,p}N(s)_{j,q}Y_{i,j} \tag{11}$$

$$u(r,s) = \sum_{i=0}^{n}\sum_{j=0}^{m} N(r)_{i,p}N(s)_{j,q}Z_{i,j} \tag{12}$$

where r and s are parameters with their values between zero and one. Here, $X_{i,j}$ and $Y_{i,j}$ are the x- and y- coordinates of the control points of the solution surface and $Z_{i,j}$ are their z-coordinates.

As it is noted, in the equations above, all of the variables are written in terms of the parameters r and s, which is similar to mapping in finite elements with the concept of the base or master element. However, calculation of the partial differentials is somehow different and needs special care. With some simple calculus the following relations can be derived:

$$\frac{\partial u}{\partial x} = \frac{\frac{\partial u}{\partial r}\frac{\partial y}{\partial s} - \frac{\partial u}{\partial s}\frac{\partial y}{\partial r}}{\frac{\partial x}{\partial r}\frac{\partial y}{\partial s} - \frac{\partial x}{\partial s}\frac{\partial y}{\partial r}} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{-\frac{\partial u}{\partial r}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial s}\frac{\partial x}{\partial r}}{\frac{\partial x}{\partial r}\frac{\partial y}{\partial s} - \frac{\partial x}{\partial s}\frac{\partial y}{\partial r}} \tag{13}$$

In addition, the Jacobian determinant $\bar{J}$ might be defined as

$$\bar{J} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{vmatrix} = \begin{vmatrix} J_{xr} & J_{yr} \\ J_{xs} & J_{ys} \end{vmatrix} = J_{xr}J_{ys} - J_{yr}J_{xs} \tag{14}$$

where

$$J_{xr} = \frac{\partial x}{\partial r} = \sum_{i=0}^{n}\sum_{j=0}^{m} N'(r)_{i,p} N(s)_{j,q} X_{i,j} \tag{15}$$

$$J_{xs} = \frac{\partial x}{\partial s} = \sum_{i=0}^{n}\sum_{j=0}^{m} N(r)_{i,p} N'(s)_{j,q} X_{i,j} \, , \tag{16}$$

$$J_{yr} = \frac{\partial y}{\partial r} = \sum_{i=0}^{n}\sum_{j=0}^{m} N'(r)_{i,p} N(s)_{j,q} Y_{i,j} \tag{17}$$

and

$$J_{ys} = \frac{\partial y}{\partial s} = \sum_{i=0}^{n}\sum_{j=0}^{m} N(r)_{i,p} N'(s)_{j,q} Y_{i,j} \, . \tag{18}$$

Considering that $dxdy = \bar{J}drds$, and in the absence of the boundary terms and assuming $q = 0$, it follows

$$\Pi = \frac{1}{2}\int_{\Omega_e} \left[k_x \left(\frac{\frac{\partial u}{\partial r}J_{ys} - \frac{\partial u}{\partial s}J_{yr}}{\bar{J}} \right)^2 + k_y \left(\frac{-\frac{\partial u}{\partial r}J_{xs} + \frac{\partial u}{\partial s}J_{xr}}{\bar{J}} \right)^2 + pu^2 \right] \bar{J}drds. \tag{19}$$

By substitution of (10)-(18) into (19), and differentiating it with respect to $Z_{i,j}$ the matrix of coefficients and a linear system of equations are obtained.

4. SOME EXPERIENCES WITH THE METHOD

To study performance of the method, as an example, solution of the Laplace equation over a square domain is here considered

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \ ; \qquad 0 \le x \le 1 \ \ , \ \ 0 \le y \le 1 \tag{20}$$

This equation is a special case of Equation (4), with $k_x = k_y = 1$ and $p = q = 0$. The boundary conditions are assumed to be

$$\begin{aligned} &u(0,y)=0; && u=(1,y)=0 \\ &u(x,0)=x(1-x); && u(x,1)=0, \end{aligned} \tag{21}$$

The exact solution of this problem is known and is given by the following infinite series [6]:

$$u(x,y)=\sum_{n=1}^{\infty}-\frac{4\sin(n\pi x)\left((-1)^n-1\right)\sinh\left(n\pi(1-y)\right)}{\sinh(n\pi)n^3\pi^3} \tag{22}$$

4.1 Comparison with the finite element method

In order to compare the performance of the method with finite elements, Equation (20) with boundary conditions as in (21) is considered. A grid of 5×5 control points (*n*=4), as depicted in Figure 2, is employed. We call this grid regular, although a couple of control points on the bottom edge are moved to meet the boundary conditions. The degree of spline basis functions is taken as *p*=2 and a uniform knot vector with *m*=7 is employed for both directions with the uniform knot vectors *U*=*V*={0,0,0,0.333,0.666,1,1,1}. The obtained solution is illustrated in Figure 3 and the results for a few specified points of the domain is shown in Table 1 and are compared with the Exact solution, which is obtained by using (22) with sufficient terms. The results of the finite element method, together with the error percentages, are also included in the Table.

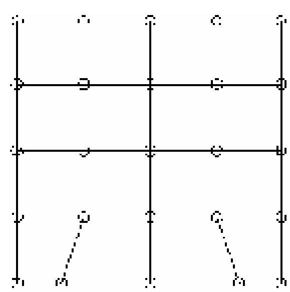


Figure 2. A grid of regular control points

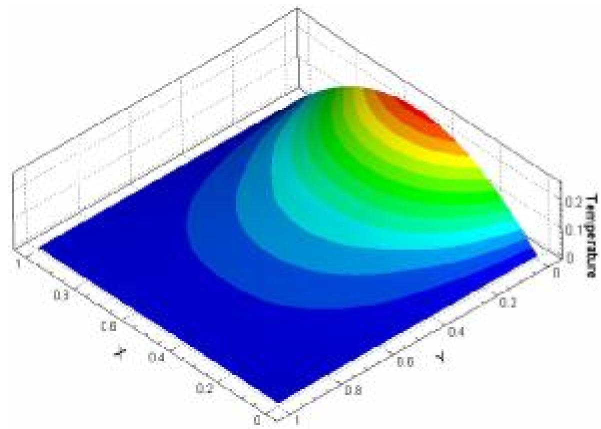
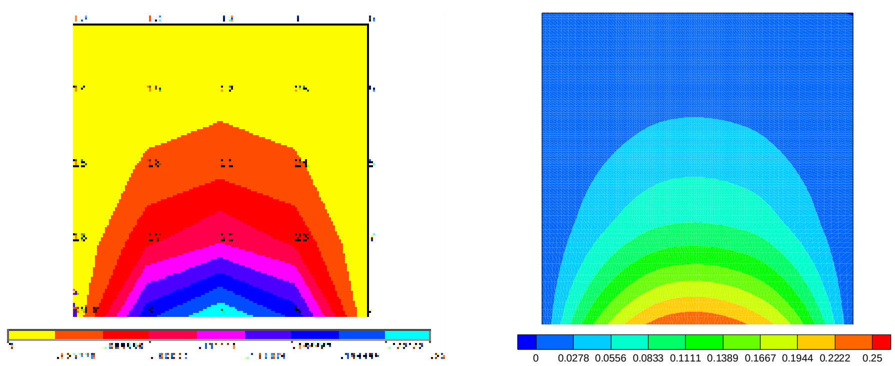


Figure 3. Isogeometrical solution with the regular control points of Figure 2

This problem is also solved by the finite element method for comparison. For this purpose, the quad element of ANSYS software is used and the obtained result is illustrated in Figure 4-(a). Figure 4-(b) shows the result obtained by the current method for the sake of



(a) (b)

Figure 4. (a) Finite element solution with 16 elements by ANSYS and (b) Isogeometrical solution with a gird of regular control points.

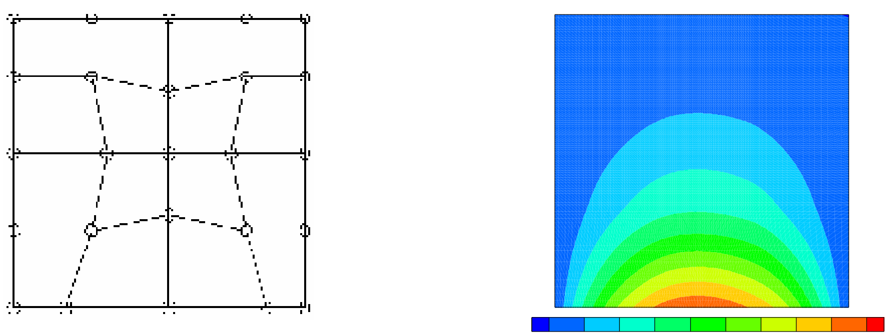
comparison. It should be noted that the number of elements is chosen in such a way that the number of equations, i.e. the size of the matrix of coefficients, is the same as the isogeometrical solution. This enables us to compare the accuracy of the results. As it is noticed from Table 1, where at some specific points the values of the unknown function with different methods are shown, better results are obtained with the current method.

Table 1. Comparison of Exact, Finite Elements and Isogeometrical solutions for a regular grid

X	Y	Exact	FE	Isogeom.	FE Err. %	Iso. Err. %
0.25000	0.25000	0.08320	0.07880	0.08156	5.28	1.97
0.50000	0.25000	0.11593	0.11211	0.11257	3.30	2.90
0.75000	0.25000	0.08320	0.07880	0.08156	5.28	1.97
0.25000	0.50000	0.03642	0.03348	0.03604	8.05	1.03
0.50000	0.50000	0.05133	0.04732	0.05197	7.81	-1.24
0.75000	0.50000	0.03642	0.03348	0.03604	8.05	1.03
0.25000	0.75000	0.01373	0.01227	0.01343	10.62	2.16
0.50000	0.75000	0.01940	0.01735	0.01882	10.55	2.99
0.75000	0.75000	0.01373	0.01227	0.01343	10.62	2.16
			Absolute error average =		7.73	1.93

4.2 Effect of irregularity in the grid of control points

To study the effect of irregularities in the grid of control points, the same problem is solved with two different grids. In the first one a so named semi-regular grid of control points, as



(a) (b)

Figure 5. (a) Semi-regular grid of control points; (b) Non-regular grid of control points

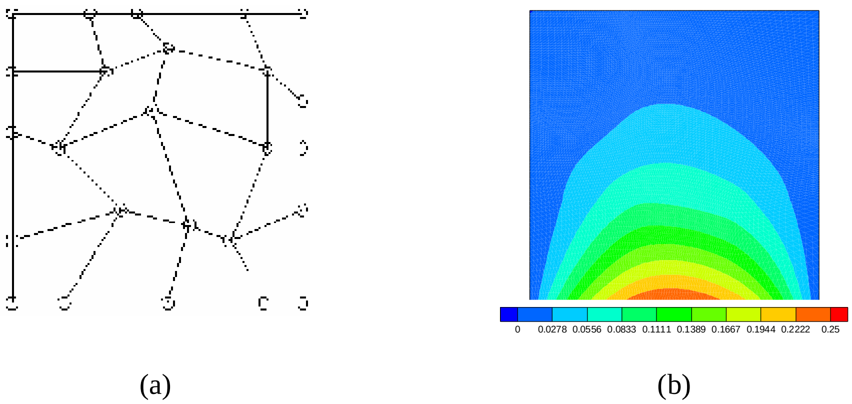


Figure 6. Isogeometrical solution with (a) Semi-regular grid and (b) Non-regular grid of control points

Shown in Figure 5-(a), is considered. The knot vectors and degrees of the basis functions are chosen as before. The answers at a few points are tabulated in Table 2 and the isothermal contours are illustrated in Figure 5-(b). As it is observed, the irregularity has very little effect on the obtained results.

Table 2. Accuracy of results with semi-regular grid of control points

X	Y	Exact	FE	Isogeom.	FE Err. %	Iso. Err. %
0.25000	0.25000	0.08320	0.07880	0.08252	5.28	0.82
0.50000	0.25000	0.11593	0.11211	0.11452	3.30	1.21
0.75000	0.25000	0.08320	0.07880	0.08252	5.28	0.82
0.25000	0.50000	0.03642	0.03348	0.03531	8.05	3.02
0.50000	0.50000	0.05133	0.04732	0.05221	7.81	-1.72
0.75000	0.50000	0.03642	0.03348	0.03531	8.05	3.02
0.25000	0.75000	0.01373	0.01227	0.01308	10.62	4.69
0.50000	0.75000	0.01940	0.01735	0.01869	10.55	3.64
0.75000	0.75000	0.01373	0.01227	0.01308	10.62	4.69
				Absolute error average =	7.73	2.62

In the next experiences, a grid of control points with more irregularities, as is illustrated in

Figure 6-(a), is taken into account. This may occur in the application of the method in some problems such as large deformation analysis of solids when solved in a Lagrangian frame. Similar to the finite element method, larger errors are normally expected with irregular meshes. However, as it can be noticed from Table 3 and Figure 6-(b), the grid irregularity has a limited effect on the accuracy of the obtained results.

Table 3. Accuracy of results with irregular grid of control points

X	**Y**	**Exact**	**FE**	**Isogeom.**	**FE Err. %**	**Iso. Err. %**
0.25000	0.25000	0.08320	0.07880	0.08109	5.28	2.54
0.50000	0.25000	0.11593	0.11211	0.11235	3.30	3.09
0.75000	0.25000	0.08320	0.07880	0.07956	5.28	4.38
0.25000	0.50000	0.03642	0.03348	0.03587	8.05	1.50
0.50000	0.50000	0.05133	0.04732	0.05113	7.81	0.38
0.75000	0.50000	0.03642	0.03348	0.03474	8.05	4.60
0.25000	0.75000	0.01373	0.01227	0.01441	10.62	-4.93
0.50000	0.75000	0.01940	0.01735	0.02005	10.55	-3.38
0.75000	0.75000	0.01373	0.01227	0.01422	10.62	-3.59
			Absolute error average =		7.73	3.15

4.3 Effect of the knot vector

In order to study the effect of the knot vector on the isogeometrical solution, the regular grid of control points of Section 4.1, as illustrated in Figure 2, is again considered. The degree of the basis functions are as before. Instead of the uniform knot vector in both directions, the following knot vectors are here employed: $U = \{0,0,0,0.4,0.8,1,1,1\}$ and $V = \{0,0,0,0.2,0.5,1,1,1\}$.

The obtained results are shown in Table 4. As is noticed, having non-uniform knot vectors with different knot spans in horizontal and vertical directions does not have a negative influence on the solution.

Table 4. Comparison of FE and exact with the Isogeometrical solution with a regular grid and non-uniform knot vectors

X Coord	Y Coord	Exact Sol.	FE Sol.	Isogeom.	%FE Err	%Iso. Err
0.25000	0.25000	0.08320	0.07880	0.08086	5.28	2.81
0.50000	0.25000	0.11593	0.11211	0.11166	3.30	3.69
0.75000	0.25000	0.08320	0.07880	0.08191	5.28	1.55
0.25000	0.50000	0.03642	0.03348	0.03554	8.05	2.41
0.50000	0.50000	0.05133	0.04732	0.05103	7.81	0.57
0.75000	0.50000	0.03642	0.03348	0.03588	8.05	1.47
0.25000	0.75000	0.01373	0.01227	0.01335	10.62	2.77
0.50000	0.75000	0.01940	0.01735	0.01883	10.55	2.95
0.75000	0.75000	0.01373	0.01227	0.01346	10.62	1.98
				Absolute error average =	7.73	2.24

4.4 Non-rectangular domain

To study the performance of the method when applied to domains with more general shapes, the Laplace equation is considered again on the assumed domain as illustrated in Figure 7. The boundary condition are considered as: $u(x,0)=x(1-x)$, and $u=0$ on other boundaries.

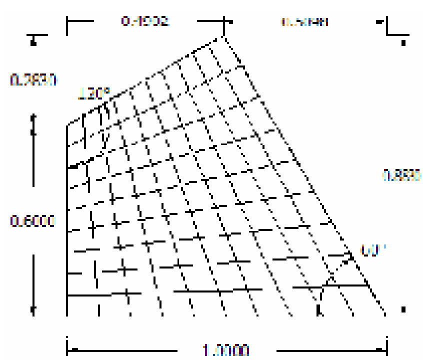
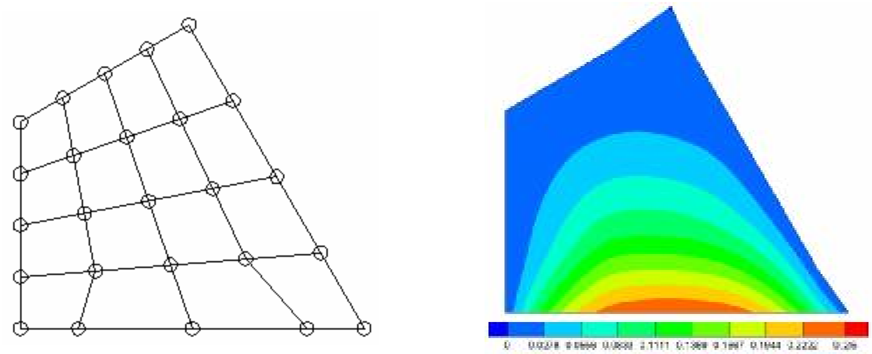


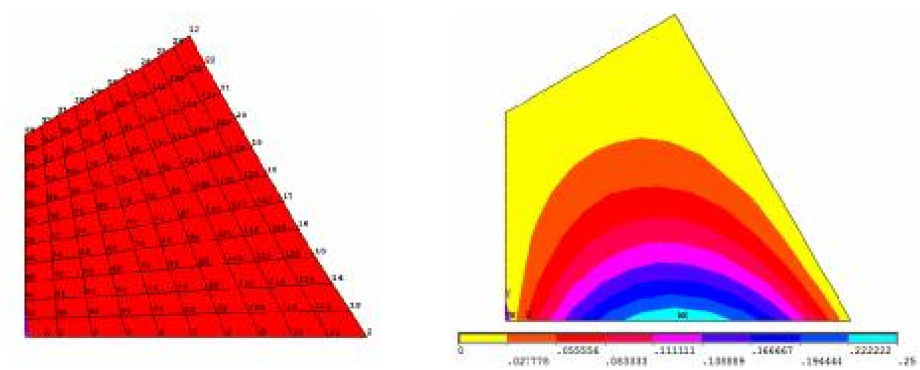
Figure 7. Problem domain

The used control net together with the contours of the solution is shown in Figure 8. This problem is also solved with ANSYS software by using the finite element mesh of Figure 9-(a) and the solution is illustrated in Figure 9-(b). It is interesting to note that here the matrix of coefficients in isogeometrical analysis is a 9 by 9 matrix and much smaller in comparison with the finite element solution.



(a) (b)

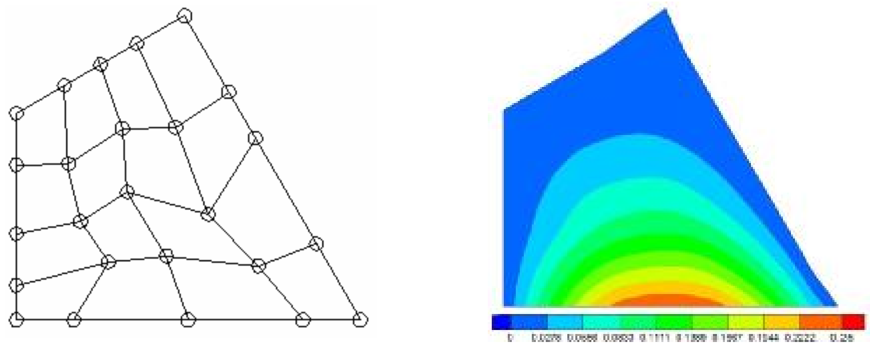
Figure 8. (a) Control net (b) Isogeometrical solution



(a) (b)

Figure 9. (a) Finite element mesh (b) Temperature contours by ANSYS

Again, to see the effect of having irregularities in the control net, this problem is solved with the net of control points of Figure 10-(a) and the obtained result is shown in Figure 10-(b). As can be observed, this irregularity has little effect on the obtained result.



(a) (b)

Figure 10. (a) non-regular control net (b) Isogeometrical solution

5. CONCLUSIONS

According to this research, it seems that the isogeometrical analysis potentially has the capability to substitute the finite element and meshfree methods. By this method the boundaries can be defined with more precision and the boundary conditions can be satisfied all along the boundary, not just at a few descretization boundary points. When applied to the solution of Laplace equation, better results in comparison with the finite element method are obtained. Furthermore, the results are not sensitive to the position of control points as well as the knot vectors. Therefore, this method is quite suitable for an adaptive solution and applicable to finite strain problems with geometrical nonlinearity. More research is needed to get a better understanding of the performance of the method in its application to multivariable partial differential equations encountered in different fields of science and engineering.

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