

*Technical Note*

**BUCKLING AND BENDING ANALYSES OF CANTILEVER  
CARBON NANOTUBES USING THE EULER-BERNOULLI BEAM  
THEORY BASED ON NON-LOCAL CONTINUUM MODEL**

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**ABSTRACT**

Euler-Bernoulli beam model based non-local elasticity theory is developed for the static and buckling analysis of cantilever carbon nanotubes (CNTs). The size effect is taken into consideration using the Eringen's non-local elasticity theory. The derivation of governing equation of bending and buckling from the shear and moment resultants of the beam and stress-strain relationship of the one-dimensional non-local elasticity model is presented. Buckling and deflection values of CNTs are obtained and presented in graphical form. Numerical results are presented to show the small-scale effect on bending of CNTs.

**Keywords:** Carbon nanotubes; non-local elasticity theory; static analysis; buckling; differential quadrature

**1. INTRODUCTION**

The concept of carbon nanotubes was first introduced in 1991 by Iijima [1] in Japan. It has many interesting mechanical properties in engineering applications [2]. Thus, the studies of mechanical behaviors of carbon nanotubes have been attracting more and more attentions of scientists in the world and also have become a new research area of applied mechanics [3, 4]. In the continuum approach, carbon nanotubes are modeled as elastic beams such as Timoshenko, Euler-Bernoulli, shells and elastic rod. Continuum models are more practical approach for nanoscale system [5-8]. In the present work, the consistent governing equations for the beam model for cantilever CNTs are derived for bending and buckling analysis. The results for static bending and critical buckling are obtained by using the differential quadrature method.

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## 2. NON-LOCAL ELASTICITY THEORY OF ERINGEN'S

It is known that, the stress state of any body at a point  $x$  is related to the strain state at the same point  $x$  in the classical elasticity. Namely, the constitutive equations of classical (macroscopic) elasticity are an algebraic relationship between the stress and strain components. But this theory is not conflict the atomic theory of lattice dynamics and experimental observation of phonon dispersion. As stated by Eringen [9], the linear theory of non-local elasticity leads to a set of integropartial differential equations for the displacements field for homogeneous, isotropic bodies. According to the non-local elasticity theory of Eringen's, the stress at any reference point in the body depends not only on the strains at this point but also on strains at all points of the body. This definition of the Eringen's non-local elasticity is based on the atomic theory of lattice dynamics and some experimental observations on phonon dispersion. In this theory, the fundamental equations involve spatial integrals which represent weighted averages of the contributions of related strain tensor at the related point in the body. Thus theory introduces the small length scale effect through a spatial integral constitutive relation. For Homogenous and isotropic elastic solids, the linear theory of non-local elasticity is described by the following equations [9]:

$$\sigma_{kl,l} + \rho(f_l - \frac{\partial^2 u_l}{\partial t^2}) = 0, \quad (1)$$

$$\sigma_{kl}(x) = \int_V \alpha(|x - x'|, \chi) \tau_{kl}(x') dV(x'), \quad (2)$$

$$\tau_{kl}(x') = \lambda \varepsilon_{mm}(x') \delta_{kl} + 2\mu \varepsilon_{kl}(x'), \quad (3)$$

$$\varepsilon_{kl}(x') = \frac{1}{2} \left( \frac{\partial u_k(x')}{\partial x'_l} + \frac{\partial u_l(x')}{\partial x'_k} \right), \quad (4)$$

where  $\sigma_{kl}$  is the non-local stress tensor,  $\rho$  is the mass density of the body,  $f_l$  is the body (or applied) force density,  $u_l$  is the displacement vector at a reference point  $x$  in the body,  $\tau_{kl}(x')$  is the classical (Cauchy) or local stress tensor at any point  $x'$  in the body,  $\varepsilon_{kl}(x')$  is the linear strain tensor at point  $x'$  in the body,  $t$  is denoted the time,  $V$  is the volume occupied by the elastic body,  $\alpha|x - x'|$  is the distance in Euclidean form,  $\lambda$  and  $\mu$  are the Lamé constants. The non-local kernel  $\alpha|x - x'|$  defines as the impact of the strain at the point  $x'$  on the stress at the point  $x$  in the elastic body. The value of  $\chi$  depends on the ratio  $(e_0 a / l)$  which is material constant. The value  $a$  depends on the internal (granular distance, lattice parameter, distance between C-C bonds as molecular diameters) and external characteristics lengths (crack length or wave length) and  $e_0$  is a constant appropriate to each material for

adjusting the model to match reliable results by experiments or some other theories. If  $\alpha|x|$  takes on a Green function of a linear differential operator given as

$$\Re\alpha(|x' - x|) = \delta(|x' - x|) \quad (5)$$

The non-local constitutive relation given by Eq. (2) is reduced to the differential equation

$$\Re\sigma_{kl} = \tau_{kl} \quad (6)$$

Furthermore the integro-partial differential equation given by Eq. (1) is also reduced to the following partial differential equation

$$\tau_{kl,l} + \Re(f_l - \rho \ddot{u}_k) = 0 \quad (7)$$

Eringen [9] proposed a non-local model for this linear differential operator given as

$$\Re = 1 - (e_0 a)^2 \nabla^2 = 0 \quad (8)$$

where  $\nabla^2$  is the Laplacian. Consequently, the constitutive relations can be written as

$$\left[1 - (e_0 a)^2 \nabla^2\right] \sigma_{kl} = \tau_{kl} \quad (9)$$

### 3. NON-LOCAL EQUATIONS OF CARBON NANOTUBES

It is accepted that the some mechanical properties such as vibration, bending and buckling of the beam like micro structures based on non-local elasticity theory are entirely different from their counterparts based on the classical (macro) beam theory. Thus the theory based on size dependent non-local elasticity theory can serve as a more reasonable and proper approach in mechanical modeling of micro and nano sized components of nano mechanical devices. The non-local theory of elasticity proposed by Eringen [9] has been widely used in the past five years in many nano mechanical problems including dislocation, crack, wave propagation, vibration analysis of nanobeams, nanotubes, carbon nanotubes, and microtubules. For carbon nanotubes in one dimensional case, the non-local constitutive relations can be written as below [5]

$$\sigma_{xx} - (e_0 a)^2 \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E \varepsilon_{xx} \quad (10)$$

Where  $\sigma_{xx}$  is the axial stress,  $\varepsilon_{xx}$  is the axial strain,  $E$  the Young modulus. Assume that the displacement of beam along y axial axis is  $w(x, t)$  in terms of spatial coordinate  $x$  and time

variable  $t$ . For transversely vibration of Carbon nanotubes, the equilibrium conditions of Euler-Bernoulli beam can be written as

$$\frac{\partial^2 M(x,t)}{\partial x^2} = \rho A \frac{\partial^2 w(x,t)}{\partial t^2} \quad (11)$$

According to the linear theory of Euler-Bernoulli beam, the strain-displacements and the moment are given by

$$\varepsilon = -y \frac{\partial^2 w(x,t)}{\partial x^2}, \quad M(x,t) = \int_A y \sigma dA \quad (12, 13)$$

Multiplying on both sides of Eq. (12) by  $y$  and integrating over the cross-section area of the beam, we obtain

$$\int_A \sigma_{xx} y dA - (e_0 a)^2 \int_A y \frac{\partial^2 \sigma}{\partial x^2} dA - \int_A E y \varepsilon dA = 0 \quad (14)$$

After some mathematical manipulations, we have

$$M(x,t) - (e_0 a)^2 \frac{\partial^2 M(x,t)}{\partial x^2} + EI \frac{\partial^2 w(x,t)}{\partial x^2} = 0 \quad (15)$$

By performing the differentiating of this equation with respect to the variable  $x$  twice we have

$$\frac{\partial^2 M(x,t)}{\partial x^2} - (e_0 a)^2 \frac{\partial^4 M(x,t)}{\partial x^4} + EI \frac{\partial^4 w(x,t)}{\partial x^4} = 0 \quad (16)$$

Finally, the following governing non-local equation of carbon nanotubes based on Euler-Bernoulli beam theory can be obtained:

$$EI \frac{\partial^4 w(x,t)}{\partial x^4} + \rho A \frac{\partial^2 w(x,t)}{\partial t^2} - (e_0 a)^2 \rho A \frac{\partial^4 w(x,t)}{\partial x^2 \partial t^2} = 0 \quad (17)$$

It is exactly seen from the Eq. (17) that the local Euler-Bernoulli beam theory is obtained when the parameter  $e_0$  is set identically to zero. In this case, non-local bending moment can be written as [2]

$$M(x) = \frac{EI}{L^2} \left[ -\frac{\partial^2 w(x)}{\partial x^2} - (e_0 a)^2 \rho A \frac{L^2}{EI} \omega^2 w(x) \right] \quad (18)$$

If we consider the Euler-Bernoulli beam subjected to a distributed load, the general non-

local equation for this case is given below

$$EI \frac{\partial^4 w(x)}{\partial x^4} + (e_0 a)^2 \frac{\partial^2 q(x)}{\partial x^2} = q(x) \quad (19)$$

By using the relation between the load and deflection, that is

$$EI \frac{\partial^4 w(x)}{\partial x^4} = q(x) \quad (20)$$

Substituting the Eq. (20) into Eq. (19) we obtain

$$EI \frac{\partial^4 w(x)}{\partial x^4} + (e_0 a)^2 \frac{\partial^6 w(x)}{\partial x^6} = q(x) \quad (21a)$$

Similarly, buckling Equation can be written as

$$EI \frac{\partial^4 w(x)}{\partial x^4} + (e_0 a)^2 \frac{\partial^6 w(x)}{\partial x^6} + P \frac{\partial^2 w(x)}{\partial x^2} - P(e_0 a)^2 \frac{\partial^3 w(x)}{\partial x^3} = 0 \quad (21b)$$

Clamped boundary conditions are considered. These are:

For clamped (C) end (at  $x=0$ )

$$w = 0 \text{ and } dw/dx = 0 \quad (22a)$$

For free (F) end (at  $x=L$ )

$$V = 0 \text{ and } M = 0 \quad (22b)$$

#### 4. DIFFERENTIAL QUADRATURE (DQ) METHOD

The advance in computational technology in the past fifty years have led to the born some new numerical methods; such as meshless methods, differential quadrature, spectral methods and discrete singular convolution methods. Differential quadrature (DQ) method is a relatively new numerical technique in applied mechanics. The method of DQ can yield accurate solutions with relatively much fewer grid points. It has been also successfully employed for different solid and fluid mechanic problems [10,11]. Unlike the DQ that uses the polynomial functions, such as power functions, Lagrange interpolated, and Legendre polynomials as the test functions, harmonic differential quadrature (HDQ) uses harmonic or

trigonometric functions as the test functions. Shu and Xue [10] proposed an explicit means of obtaining the weighting coefficients for the HDQ. When the  $f(x)$  is approximated by a Fourier series expansion in the form

$$f(x) = c_0 + \sum_{k=1}^{N/2} \left( c_k \cos \frac{k\pi x}{L} + d_k \sin \frac{k\pi x}{L} \right) \quad (23)$$

and the Lagrange interpolated trigonometric polynomials are taken as

$$h_k(x) = \frac{\sin \frac{(x-x_0)\pi}{2} \cdots \sin \frac{(x-x_{k-1})\pi}{2} \sin \frac{(x-x_{k+1})\pi}{2} \cdots \sin \frac{(x-x_N)\pi}{2}}{\sin \frac{(x_k-x_0)\pi}{2} \cdots \sin \frac{(x_k-x_{k-1})\pi}{2} \sin \frac{(x_k-x_{k+1})\pi}{2} \cdots \sin \frac{(x_k-x_N)\pi}{2}} \quad (24)$$

for  $k = 0, 1, 2, \dots, N$ . According to the HDQ, the weighting coefficients of the first-order derivatives  $A_{ij}$  for  $i \neq j$  can be obtained by using the following formula:

$$A_{ij} = \frac{(\pi/2)P(x_i)}{P(x_j) \sin[(x_i - x_j)/2]\pi}; \quad i, j = 1, 2, 3, \dots, N, \quad (25)$$

Where

$$P(x_i) = \prod_{j=1, j \neq i}^N \sin \left( \frac{x_i - x_j}{2} \pi \right); \quad \text{for } j = 1, 2, 3, \dots, N. \quad (26)$$

The weighting coefficients of the second-order derivatives  $B_{ij}$  for  $i \neq j$  can be obtained using following formula:

$$B_{ij} = A_{ij} \left[ 2 A_{ii}^{(1)} - \pi \cot \left( \frac{x_i - x_j}{2} \pi \right) \right]; \quad i, j = 1, 2, 3, \dots, N, \quad (27)$$

The weighting coefficients of the first-order and second-order derivatives  $A_{ij}^{(p)}$  for  $i = j$  are given as

$$A_{ii}^{(p)} = - \sum_{j=1, j \neq i}^N A_{ij}^{(p)}; \quad p = 1 \text{ or } 2; \text{ and for } i = 1, 2, \dots, N. \quad (28)$$

By using DQ discretization for example for bending, the Eq. (21) takes the form

$$\frac{EI}{L^4} \sum_{j=1}^N D_{ij} w_j + q_i (e_0 a)^2 \sum_{j=1}^N F_{ij} w_j = q_i \quad (29)$$

Similarly, the buckling equation can be written in same form given above.

## 5. NUMERICAL RESULTS

In this section, several examples of CNTs having clamped boundary conditions are examined. For this purpose, some numerical results in the forms of graphs are presented using the method of DQ. The material and geometric constants of CNTs are given in Table 1. The symbol C-F, represents the CNT having clamped edge at  $x=0$  and free edge at  $x=L$ . To determine the accuracy and convergence of the present method for deflection analysis of CNTs, numerical experimentation was carried out by varying the number of grid points  $N$ .

Table 1: Material and geometric values of CNTs

Parameter	Value
$E$	$1 \times 10^{12} \text{ N/m}^2$
$I$	$491 \times 10^{-36} \text{ m}^4$
$\rho$	$2300 \text{ kg/m}^3$
$L$	$10^{-8} \text{ m}$

The obtained results of non-dimensional static deflections are computed for different number of terms in  $N$  is shown in Table 2. Exact analytical solution [3] is also given for comparison. Excellent agreement has been achieved between the present results and the results obtained by analytical formula given by Reddy [3]. It is seen from this table that when the grid point numbers reaches  $N=7$  the present method gives accurate predictions for the deflections. Figures 1-3 show the displacement and bending moment along the length of CNTs for different value of non-local parameters. It can be seen that the effect of non-local parameter on the deflection and bending moment is significant. In general, the non-local parameter result in an increase of the transverse deflection and bending moment of a CNT under uniformly distributed load. For the case of bending moment, the non-local parameter is generally insignificant. It is observed that when the non-local parameter is greater than 6nm, the deflection values becomes negative.

Table 2: Comparison of non-dimensional maximum deflection ( $w \times EI / qL^4$ ) of C-F carbon nanotubes under uniformly distributed loading

$e_0 a/L$	Analytical result [3]	Present DQ solutions				
		N=5	N=7	N=9	N=11	N=13
0.0	0.1250	0.12518	0.12504	0.12501	0.12501	0.12501
0.20	0.0250	0.02511	0.02507	0.02506	0.02506	0.02506

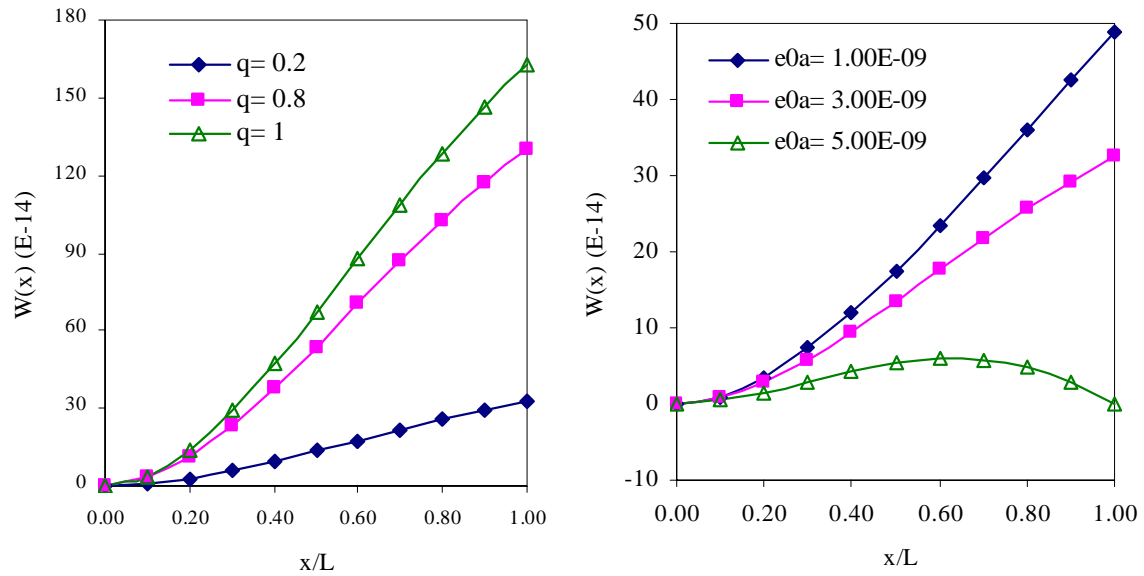


Figure 1. Deflection of C-F carbon nanotubes a)  $e_0a=0.3 \times 10^{-9}$  m b)  $q=0.2$  N/m

Critical buckling loads are obtained and results presented in Figures 4-6 for different parameters. In general, buckling loads are decrease with the increasing value of non-local parameter.

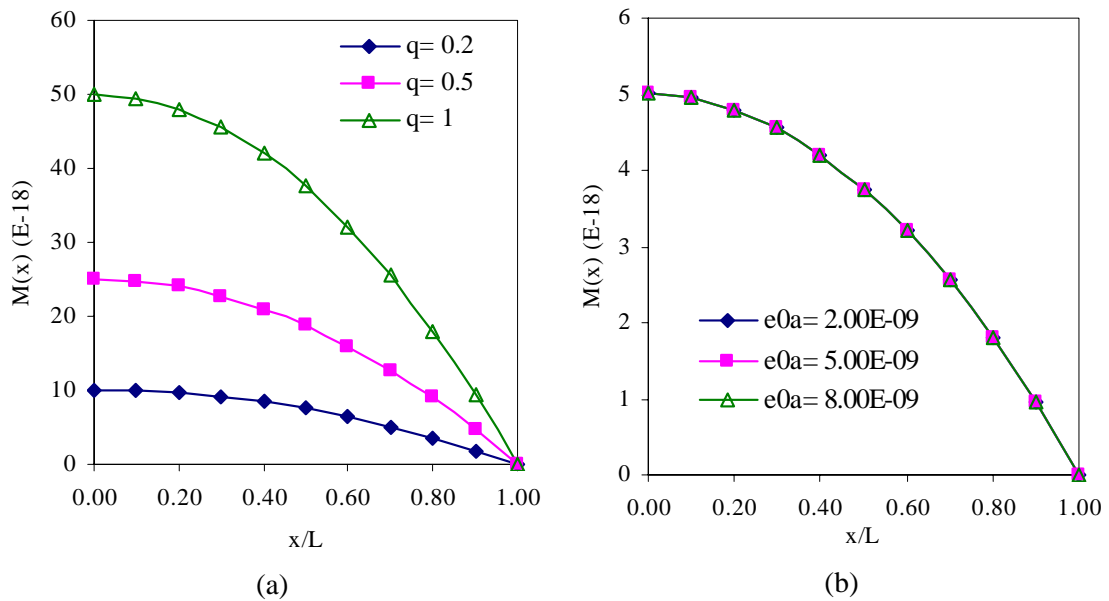


Figure 2. Bending moment of C-F carbon nanotubes a)  $e_0a=0.5 \times 10^{-9}$  m b)  $q=0.1$  N/m



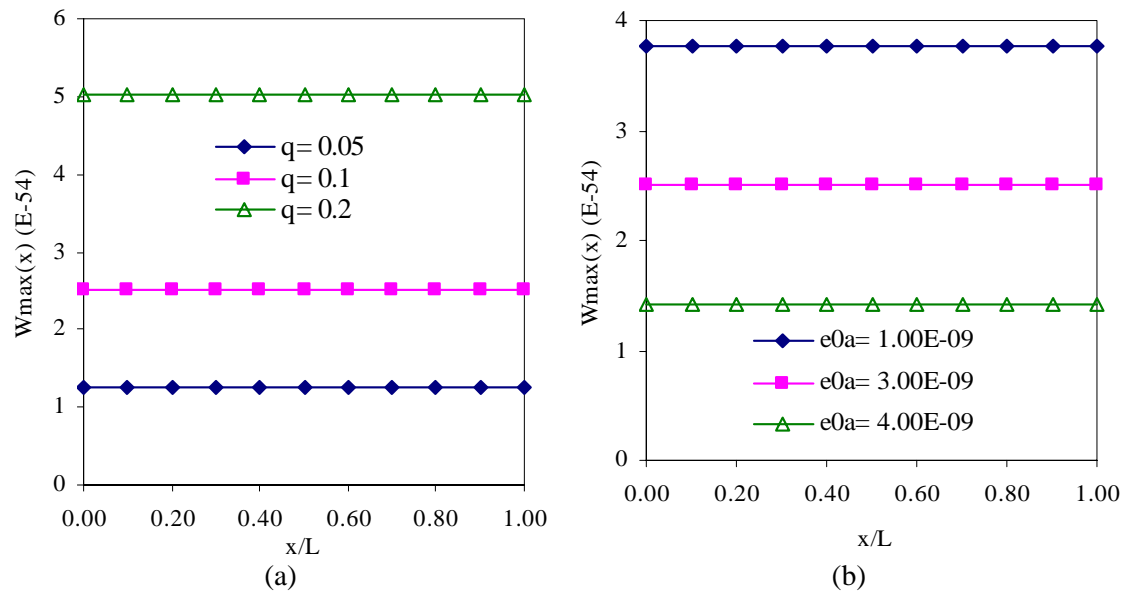


Figure 3. Maximum deflection of C-F carbon nanotubes a)  $e_0a=0.3 \times 10^{-9}$  m b)  $q=0.1$  N/m

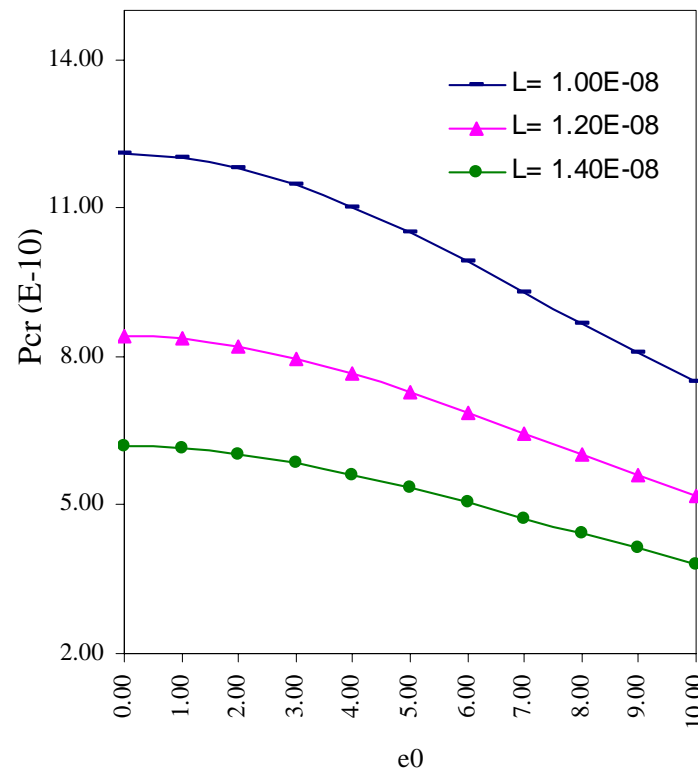


Figure 4. Variation of critical buckling load with different non-local parameter

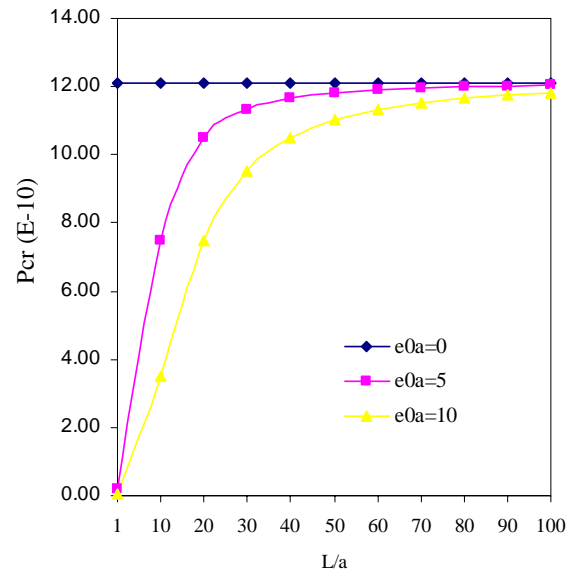


Figure 5. Variation of critical buckling load with length-to-material ratio

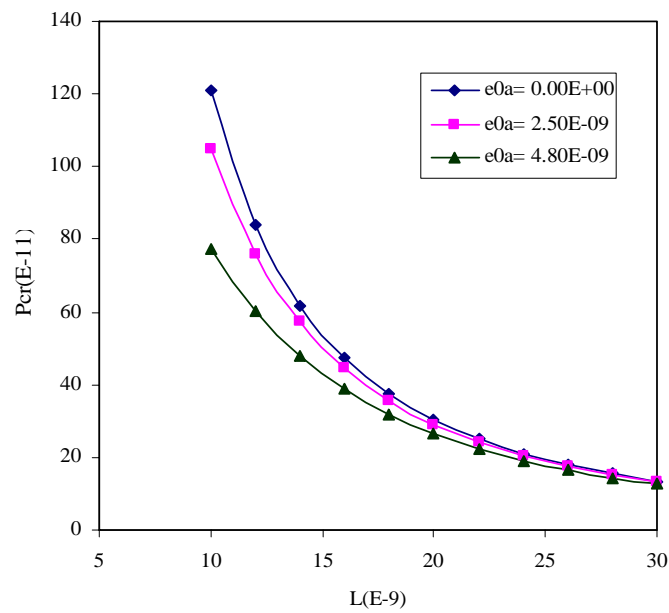


Figure 6. Variation of critical buckling load with length

## 6. CONCLUDING REMARKS

Static and buckling behavior of single walled carbon nanotubes is investigated in this paper. The numerical results show that the effect of non-local parameter is significant on static and buckling behaviour of CNTs. The method is suitable for the problem considered due to its

generality, simplicity, and potential for further development.

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## REFERENCES

1. Iijima S. Helical microtubules of graphitic carbon, *Nature*, **354**(1991) 56-8.
2. Wang Q, Liew KM. Application of non-local continuum mechanics to static analysis of micro- and nano-structures, *Physics Letters A*, **363** (2007)236-42.
3. Reddy JN. Non-local theories for bending, buckling and vibration of beams, *International Journal of Engineering Sciences*, **45**(2007) 288-307.
4. Reddy JN, Pang SD. Non-local continuum theories of beams for the analysis of carbon nanotubes, *Journal of Applied Physics*, **103** (2008) 3511-6.
5. Reddy JN, Wang CM. A bending, buckling and frequency relationships between the Euler-Bernoulli and Timoshenko non-local beam theories, *Asian Journal of Civil Engineering (Building and Housing)*, **10**(2009) 265-82.
6. Artan R, Tepe A. The initial values method for buckling non-local bars with application in nanotechnology, *European Journal of Mechanics A/Solids*, **27**(2008) 469-77.
7. Civalek Ö, Akgöz B. Static analysis of single walled carbon nanotubes (swcnt) based on Eringen's non-local elasticity theory, *International Journal of Engineering and Applied Sciences (IJEAS)*, **1**(2009) 47-56.
8. Demir Ç, Civalek Ö, Akgöz B. Free vibration analysis of carbon nanotubes based on shear deformable beam theory by discrete singular convolution technique, *Mathematical and Computational Applications*, **15**(2010)57-65.
9. Eringen AC. On differential equations of non-local elasticity and solutions of screw dislocation and surface waves, *Journal of Applied Physics*, **54** (1983) 4703-10.
10. Shu C, Xue H. Explicit computations of weighting coefficients in the harmonic differential quadrature, *Journal of Sound and Vibration*, **204**(1997)549-55.
11. Civalek Ö. Application of differential quadrature and harmonic differential quadrature for buckling analysis of thin isotropic plates and elastic columns, *Engineering Structures*, **26**(2004)171-86.