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EFFICIENT SOLUTION OF DIFFERENTIAL EQUATIONS FOR LINEAR AND NON-LINEAR ANALYSIS OF STRUCTURES

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ABSTRACT

In this paper using the eigenvalues and eigenvectors of symmetric and asymmetric of tri-diagonal matrices with infinite dimension, inverse of these matrices are found. Moreover numerical method of finite difference is used for numerical modeling of engineering problems. Many engineering problems modeled by numerical methods result in solution of linear systems. Many of these linear systems are in tri-diagonal form. The inverse of this kinds of matrices have many applications. In this paper, response of linear and nonlinear structures resulting from different types of arbitrary dynamics and statics analyses are presented. The method of this paper provides closed form solutions for the analysis of structures under different static and seismic loadings.

Keywords: Static and dynamic response of structures; eigenvalues and eigenvectors; tri-diagonal matrices; seismic loading.

1. INTRODUCTION

For obtaining the dynamics response of structure, first it is necessary to solve a differential equation of the following form:

$$\mathbf{M}\,\ddot{\mathbf{U}}(t) + \mathbf{C}\,\dot{\mathbf{U}}(t) + \mathbf{K}\,\mathbf{U}(t) = \mathbf{P}(t) \tag{1}$$

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The most important point in the solution of the above equation is the term **P**(*t*). Since depending the function representing the loading of the structure, the particular solution of Eq. (1) will be different and finding a closed form solution for all the loads will not be possible. Though for special types of loading, such as harmonic, impulsive and periodic loading one can provide closed form solutions, however, for more complex ones like seismic loading, the solution of Eq. (1) requires numerical time step methods [1-2].

For numerical solution of dynamic problems, many approaches are available. For these methods two aspects are important. First the numerical algorithm, and second the mathematical aspect in relation with convergence, accuracy and stability of the method. In the following some methods for direct integration are briefly introduced.

One of the numerical methods is the central difference approach and for equal time step $\Delta t_i = \Delta t$, velocity and acceleration can be expressed in terms of displacements in the range of specified time. In this method, if small time steps are chosen, then more accurate results will be obtained. However, this will require more computational time [1-3].

Another method is the Houbolt integration method that is identical to central difference method. Similar to the central difference method, the velocity and acceleration in each instance depends on the displacements.

The other method is the Wilson θ method. In this method acceleration between i and $i+\theta$ points is assumed as a linear function. Other numerical methods are also available such as Newmarks integration and Duhamel's integral approaches.

In dynamics problems all these method have many applications, however in all of them analysis is performed in time steps. In fact, if displacement of structure in the time i and before are calculated, then we can calculate the displacement in the time $i+1$. Thus none of these methods can calculate the displacement of structure in the time step i without calculation at all the previous time steps [1-3].

In this paper, a method is proposed, by which it is possible to calculate the structural response under any kind of arbitrary loading. The important point about this method is that the displacement of structure calculating in the time step i is possible without calculating the displacement in the previous time steps. Thus it will be easy to calculate the response of any kind of structure under arbitrary loading using the presented approach.

2. FINITE DIFFERENCE METHOD

One of methods for solution of the differential equation is the finite difference approach. The equations of this method are based on the coefficients of the Tailor's series. Here the response derivatives are expressed in terms of velocity and acceleration, which is an approximation of the displacement. In this paper solution of the dynamic and static equilibrium equations are required which are obtained utilizing the central finite difference method [4].

Here, finite difference equations are written for solution of the following differential equation:

$$y''(x) + P(x)y'(x) + Q(x)y(x) = F(x) \qquad x \in [a,b] \tag{2}$$

If the entire range of the load is divided into *n* parts, the length of each segment will be *h*. The following indexing is considered for ease:

$$y(x_i) = y_i\ ,\ \ P(x_i) = P_i\ ,\ Q(x_i) = Q_i\ ,\ F(x_i) = F_i \tag{3}$$

The central finite difference equation for differential Eq. (2) after reforming will be as [4]:

$$(1+\frac{h}{2}p_i)\,y_{i+1} + (h^2 Q_i - 2)\,y_i + (1-\frac{h}{2}p_i)\,y_{i-1} = h^2 F_i \tag{4}$$

Considering the above equation, it is possible to present the following equation for the finite difference expansion of Eq. (1):

$$(1+\frac{h}{2}2\xi_n\omega_n)\,y_{i+1} + (h^2\omega_n^2 - 2)\,y_i + (1-\frac{h}{2}2\xi_n\omega_n)\,y_{i-1} = h^2\frac{P_i}{M_n} \tag{5}$$

As we can see in Eq. (5), both $2\xi_n\omega_n = p_i$ and $\omega_n^2 = Q_i$ are constant, that do not vary with time and $\frac{P_i}{M_n} = F_i$ changes in time. It should be mentioned that in free vibration $\frac{P_i}{M_n} = F_i = 0$. Then matrix form of Eq. (5) in the time range of [a,b] will be presented in the next section, where the time range is divided into *n* parts. It should be mentioned that in static only the stiffness matrix will be present as a tri-diagonal form, and the presented method will be applicable. For dynamic analysis of structure for both damped and undamped cases the method is also applicable as will be illustrated in the coming sections.

2.1. Undamped structures

Ignoring the damping and primary velocity, Eq. (5) in an extended form will be as:

$$\begin{bmatrix} \left(\frac{b\omega_n}{n}\right)^2-2 & 1 & 0 & \cdots & 0 \\ 1 & \left(\frac{b\omega_n}{n}\right)^2-2 & 1 & \ddots & 0 \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \left(\frac{b\omega_n}{n}\right)^2-2 & 1 \\ 0 & \cdots & 0 & 1 & \left(\frac{b\omega_n}{n}\right)^2-2 \end{bmatrix} . \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} \left(\frac{b}{n}\right)^2 \frac{P_1}{M_n} - y_0 \\ \left(\frac{b}{n}\right)^2 \frac{P_2}{M_n} \\ \vdots \\ \left(\frac{b}{n}\right)^2 \frac{P_{n-1}}{M_n} \\ \left(\frac{b}{n}\right)^2 \frac{P_n}{M_n} - y_{n+1} \end{bmatrix} \tag{6}$$

The coefficient matrix is a tri-diagonal and symmetric matrix. Now if the inverse of coefficient matrix is calculated then displacement can easily be obtained in each instant. The inverse can be calculated using the eigenvalues and eigenvectors of coefficient matrix. For A is a considered matrix, the eigenvector matrix is Q and its eigenvalue diagonal matrix is Λ, then its inverse of A can be calculated utilizing Eq. (7) as:

$$A^{-1} = Q\ \Lambda^{-1}\ Q^{-1} \tag{7}$$

In the above equation, if the dimension of the matrix is very large, then finding the inverse of Q requires a lot of computational time. Since the inverse of an eigenvalue matrix for a symmetric matrix is the same as its transpose, then the calculation becomes much easier. In the following equation the inverse of matrix A is given when it is symmetric:

$$A^{-1} = Q\ \Lambda^{-1}\ Q^{T} \tag{8}$$

In the above equation since the matrix Λ is diagonal, therefore calculation of its inverse is very easy. As we can observed before the coefficient matrix in Eq. (6) is symmetric. Thus using Eq. (8) its inverse can be calculated. Since for using this method, it is necessary to calculate the eigenvalues and eigenvectors of the coefficient matrix, thus this method will be explained in subsequent (see also [5]).

Although formulations of the eigenvalue problems are identical, with increasing the dimension of the matrices, the numerical calculation of these values becomes a challenging problem. Obviously this needs a high computer memory and spending a lot of computational time. That is why many researchers have been involved in this problem. The aim has been finding eigenvalues of larger matrices in a shorter time. For this purpose the factorization of matrices is a very useful approach [3,6,7]. Many of these factorizations are based on finding the eigenvalues of the matrices. Examples of these are Jordan, Schur, QZ, and LU

factorization [8-10].

In this method the eigenvalues and eigenvectors of the coefficient matrix shown in Eq. (6), with an arbitrary dimension, are calculated without the need for solution of equations with large size. Wen-Chyuan Yueh has presented equation for calculating tri-diagonal matrices which is similar to Eq. (9) [11].

$$A = \begin{bmatrix} -\alpha + b & c & 0 & \cdots & 0 \\ a & b & c & \ddots & 0 \\ 0 & a & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & b & c \\ 0 & \cdots & 0 & a & -\beta + b \end{bmatrix}_{n \times n} \tag{9}$$

In special case when $\alpha = \beta = 0$, the eigenvalues of matrix A in Eq. (9) is obtained as in Eq. (10) and the eigenvectors are obtained from Eq. (11) as:

$$\lambda_k = b + 2\sqrt{a\,c}\ Cos[\frac{k\pi}{n+1}], \quad k = 1,2,...,n, \tag{10}$$

$$u_j^{(k)} = \rho^{j-1}\, Sin[\frac{k\,j\pi}{n+1}], \quad j = 1,2,...,n, \tag{11}$$

Using the above equations calculation of the eigenvalues and eigenvectors for Eq. (6) is feasible. Then with the aid of Eq. (8) the inverse of the matrix can be calculated. In what follows by multiplication of the inverse matrix in loading vector, the structure response will be calculated. In Eq. (11), we have $\rho = \sqrt{\frac{a}{c}}$. Λ is the eigenvalues matrix that is diagonal; also λ_k are the entries of the main diagonal. Q is the eigenvectors matrix with $u_j^{(k)}$ being the jth row. Since the norm of each vector in Eq. (11) is not unity, so for using Eq. (8) it is necessary to multiply the inverse of eigenvector norm by the eigenvector matrix. As in Eq. (8) eigenvector matrix is used two times, so the total equation should be multiplied by the inverse of eigenvector norm to the power two by the eigenvector matrix, as is shown in the following:

$$A^{-1} = \frac{1}{|Q_i|^2} Q\ \Lambda^{-1}\ Q^T \tag{12}$$

Performing the multiplication in Eq. (12) and simplifying it a closed form solution is obtained for calculating the entries of the inverse matrix A as:

$$A_{cr}^{-1} = \sum_{k=1}^{n} \left(\frac{1}{\sum_{k=1}^{n} Sin\,[\frac{k\pi}{n+1}]^{2}} Sin\,[\frac{c\,k\,\pi}{n+1}]\;\; Sin\,[\frac{r\,k\,\pi}{n+1}] \frac{1}{\lambda_k} \right) \tag{13}$$

In the above equation, n is the dimension of coefficient matrix. Also r and c are the row and column of the considered entry in the inverse matrix, respectively. For example, in a matrix A is considered as in Eq. (9), these parameters will be equal to $a = 7, b = 2, c = 7$ and dimension of the matrix will be equal to 6. Using Eq. (13) the entries of the second row and fifth column of the inverse matrix will be equal to 1372/63881. This value is equal to the exact value that can be obtained by direct inversion of the matrix.

Now using Eq. (13), the inverse of the coefficient matrix A is calculated. If the result is multiplied by the load vector, then the response of structure in each instance will be obtained. In fact Eq. (14) is the result of multiplication of the rth row of the inverse of the coefficient matrix shown in Eq. (13), by loading vector. The load vector is shown in the right hand side of Eq. (6). Therefore by this method, a closed form solution is obtained for calculating the response of a single DOF undamped structure under an arbitrary loading is obtained as:

$$y_c = \sum_{r=1}^{n} \sum_{k=1}^{n} \left(\frac{\rho^{c-k}}{\sum_{k=1}^{n} Sin\,[\frac{k\pi}{n+1}]^{2}} Sin\,[\frac{c\,k\,\pi}{n+1}]\;\; Sin\,[\frac{r\,k\,\pi}{n+1}] \frac{1}{\lambda_k} P_k \right) \tag{14}$$

In Eq. (6) and Eq. (9) by substitution of the marked parameters in Eq. (14) will result in a closed form solution as Eq. (15), which is based on the characteristics of structure. It is important to know that in the case undamp structures we have $\rho = 1$.

$$y_c = \sum_{r=1}^{n} \sum_{k=1}^{n} \left(\frac{Sin\,[\frac{c\,k\,\pi}{n+1}]\;\; Sin\,[\frac{r\,k\,\pi}{n+1}]}{\sum_{k=1}^{n} Sin\,[\frac{k\pi}{n+1}]^{2}} \frac{P_k}{\left(\frac{b\,\omega_n}{n}\right)^{2} + 2\,Cos[\frac{k\pi}{n+1}] - 2} \right) \tag{15}$$

Using this equation and performing a modal analysis, one can find the response of a multi degree of freedom structure under an arbitrary loading in each. An important point about this method is that the response does not depend on the structural response in previous instances. While in aforementioned numerical methods the response is found in a step by step manner and without calculation the response in instance *n-1* the response in instance *n* cannot be

calculated.

2.2. Damped structures

When a damped structure is assumed and the structure does not have primary velocity at the beginning and at the end of the displacement range, then Eq. (5) will have the following form:

$$\begin{bmatrix} \left(\frac{b\omega_n}{n}\right)^2-2 & 1+\frac{b\xi_n\omega_n}{n} & 0 & \cdots & 0 \\ 1-\frac{b\xi_n\omega_n}{n} & \left(\frac{b\omega_n}{n}\right)^2-2 & 1+\frac{b\xi_n\omega_n}{n} & \ddots & \vdots \\ 0 & 1-\frac{b\xi_n\omega_n}{n} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \left(\frac{b\omega_n}{n}\right)^2-2 & 1+\frac{b\xi_n\omega_n}{n} \\ 0 & \cdots & 0 & 1-\frac{b\xi_n\omega_n}{n} & \left(\frac{b\omega_n}{n}\right)^2-2 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} \left(\frac{b}{n}\right)^2 \frac{P_1}{M_n} - y_0 \\ \left(\frac{b}{n}\right)^2 \frac{P_2}{M_n} \\ \vdots \\ \left(\frac{b}{n}\right)^2 \frac{P_{n-1}}{M_n} \\ \left(\frac{b}{n}\right)^2 \frac{P_n}{M_n} - y_{n+} \end{bmatrix} \tag{16}$$

The above coefficient matrix is a tri-diagonal matrix that is not symmetric. Since the matrix is not symmetric, using Eq. (8) is not permitted. However if its inverse is calculated then the response of structure in each instance can easily be obtained. Using the eigenvalues and eigenvectors of the coefficient matrix its inverse can be calculated. In this case, for calculating the inverse of the coefficient matrix Eq. (14), Eq. (12) will be in the following form:

$$A^{-1} = \frac{1}{|Q_i|^2} Q\, \Lambda^{-1}\, Q^{-1} \tag{17}$$

Performing the multiplication in Eq. (17) and simplifying it, a closed form solution is obtained for calculation of each entry of the inverse matrix. One can prove that the difference of Eq. (16) is only a coefficient. Considering this, the inverse of the coefficient matrix of Eq. (16) will be obtained as:

$$A_{cr}^{-1} = \sum_{k=1}^{n} \left(\frac{\rho^{c-r}}{\sum_{k=1}^{n} Sin\,[\frac{k\pi}{n+1}]^2}\, Sin\,[\frac{c\,k\,\pi}{n+1}]\;\; Sin\,[\frac{r\,k\,\pi}{n+1}] \frac{1}{\lambda_k} \right) \tag{18}$$

All the parameters utilized in Eq. (18) are previously defined. In Eq. (18), n is dimension

of coefficient matrix. Also r and c are row and column of entry in the inverse matrix, respectively. As an example, in the matrix of Eq. (9) if we have $a = 6, b = 2, c = 7,$ and the dimension of the matrix is assumed 6, then using Eq. (18), second row and fifth column of the inverse matrix will be 343/8762, which is equal to the exact values from the direct calculation of inverse matrix.

The inverse of coefficient matrix shown by Eq. (18) can be multiplied by the force vector shown in the right hand side of Eq. (16) to find the closed form solution of the structure. In fact Eq. (19) is the result of multiplying the rth row of inverse of coefficient matrix, shown in Eq. (18), by the force vector. Then the closed form solution for calculating response of undamped structures under the effect of any kind of arbitrary load will be obtained, as shown in the following:

$$y_c = \sum_{r=1}^{n}\sum_{k=1}^{n}\left(\frac{\rho^{c-k}}{\sum_{k=1}^{n} Sin\,[\frac{k\pi}{n+1}]^2}\, Sin\,[\frac{c\,k\,\pi}{n+1}]\;\; Sin\,[\frac{r\,k\,\pi}{n+1}]\frac{1}{\lambda_k}P_k\right) \tag{19}$$

Substituting the parameters of Eq. (16) and (9) in Eq. (19), the following closed form solution is obtained, in terms of the characteristics of the structure:

$$y_c = \sum_{r=1}^{n}\sum_{k=1}^{n}\left(\frac{\left(\sqrt{\frac{2n}{n+b\,\xi_n\,\omega_n}-1}\right)^{c-k}}{\sum_{k=1}^{n} Sin\,[\frac{k\pi}{n+1}]^2}\,\frac{Sin\,[\frac{c\,k\,\pi}{n+1}]\;\; Sin\,[\frac{r\,k\,\pi}{n+1}]}{\left(\frac{b\,\omega_n}{n}\right)^2 - 2 + 2\;Cos[\frac{k\pi}{n+1}]\sqrt{1-\left(\frac{b\,\xi_n\,\omega_n}{n}\right)^2}}P_k\right) \tag{20}$$

Using Eq. (20) the response of a damp structure under any kind of arbitrary load can be calculated. The important point about the presented method is that the response can be calculated independent of the response in the previous instances. While in the existing numerical method mentioned before, the calculations are performed in step by step and without calculating response in *n-1* instance it is impossible to obtain the response in the *n* instance. Some examples are presented for better illustration of this method.

3. NUMERICAL EXAMPLES

In this section, three illustrative examples are presented.

Example 1 (nonlinear analysis): In this example a rod with fixed ends is considereded. This rod is subjected to distributed axial load as shown in Fig. 1. The stress-strain relation is assumed to be bi-linear. Assuming the displacement and strain are small and that the load is applied slowly, using the presented method of this paper, and solving the equation with

modified Newton-Rofson method, deformation of the rod is obtained.

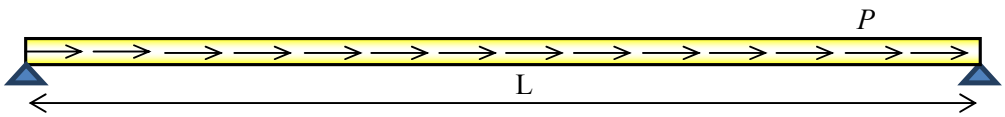


Fig. 1 A fixed ended bar element with non-linear behavior

For this purpose, a continues system is assumed in the form of a number of spring. The end of each spring is subjected to a longitudinal tensile force of magnitude P_i.

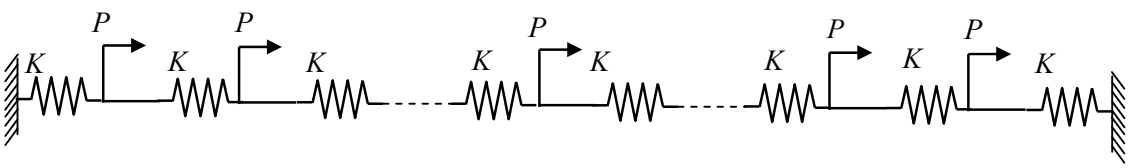


Fig. 2 The model of the bar introduced in Fig. 1

After the formation of incremental equilibrium equations, the following algebraic equations are obtained:

$$\begin{bmatrix} k_1+k_2 & -k_2 & & & \\ -k_2 & k_2+k_3 & & & \\ & & \ddots & \ddots & \\ & & \ddots & k_{n-2}+k_{n-1} & -k_{n-1} \\ & & & -k_{n-1} & k_{n-1}+k_n \end{bmatrix} \begin{bmatrix} u_{1,t} \\ u_{2,t} \\ \vdots \\ \vdots \\ u_{n,t} \end{bmatrix} = \begin{bmatrix} P_{1,t} \\ P_{2,t} \\ \vdots \\ \vdots \\ P_{n,t} \end{bmatrix} \tag{21}$$

In this example, the length of all the springs are taken identical, and thus all their stiffness k_1 to k_n will be the same. The inverse of coefficient matrix can be obtained by Eq. (13). Nodal displacement can be obtained utilizing the following closed form solution:

$$u_{c,t} = \sum_{r=1}^{n}\sum_{k=1}^{n}\left(\frac{Sin\,[\frac{c\,k\,\pi}{n+1}]\;\; Sin\,[\frac{r\,k\,\pi}{n+1}]}{\sum_{k=1}^{n} Sin\,[\frac{k\pi}{n+1}]^{2}} \frac{1}{2\left(1+Cos[\frac{k\pi}{n+1}]\right)} p_{k,t} \right) \tag{22}$$

The longitudinal nodal displacement obtained from the above equation, are shown in Fig. 3. Data considered for this example are as provided in Table 1.

Table 1: Data for the Example 1

$P(N/cm)$	$L(cm)$	ε_y	$E(N/cm^2)$	$E_T(N/cm^2)$	$A(cm^2)$
5×10^4	20	0.002	10^7	10^5	1

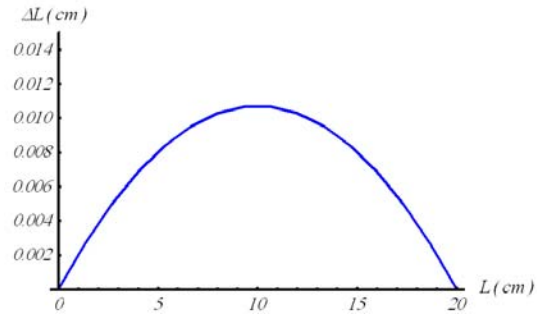


Fig. 3 The horizontal nodal displacement of Example 1

Utilizing the nodal displacements, the vector of longitudinal strain is obtained. Then it will be clear which springs are linear and which ones are nonlinear. Using the longitudinal strain, internal force of the springs can be obtained and the following incremental equilibrium equations should hold:

$$\begin{aligned} \boldsymbol{K}\ \Delta\boldsymbol{U}^{(i)} &= {}^{t+\Delta t}\boldsymbol{R} - {}^{t+\Delta t}\boldsymbol{F}^{(i-1)} \\ {}^{t+\Delta t}\boldsymbol{U}^{(i)} &= {}^{t+\Delta t}\boldsymbol{U}^{(i-1)} + \Delta\boldsymbol{U}^{(i)} \end{aligned} \tag{23}$$

Since the modified Newton-Rofson solution is used, $\boldsymbol{K}$ in Eq. (23) does not change in each iteration. The inverse of $\boldsymbol{K}$ can be obtained by Eq. (13). Thus it is clear that in Eq. (21) only the part $p_{k,t}$ will change in each iteration. In Eq. (23), ${}^{t+\Delta t}R$ is the externally applied nodal point force in this configuration at time $t+\Delta t$ and ${}^{t+\Delta t}\boldsymbol{F}^{(i-1)}$ is the vector of the nodal point forces corresponding to the element stresses in the configuration at the time $t+\Delta t$ and $(i-1)$ iteration. The numerical result for the first spring is illustrated in Fig. 4.

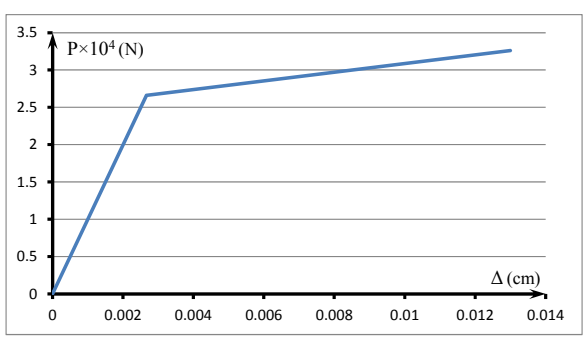


Fig. 4 The force-displacement relationship of the spring in Example 1

Example 2: In this example, the response of an *n*-story shear building with the floors being considered as rigid is obtained, Fig. 5. This structure is subjected to explosion loading in a distance relatively far from structure. If this structure is assumed as an *n*-story shear building with the stiffness each story being k_i and lumped mass of each story is m_i, then the stiffness matrix and mass matrix are as:

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & & & \\ -k_2 & k_2 + k_3 & & & \\ & & \ddots & \ddots & \\ & & \ddots & k_{n-1} + k_n & -k_n \\ & & & -k_n & k_n \end{bmatrix} \tag{24}$$

and

$$\mathbf{M} = \begin{bmatrix} m_1 & & & & \\ & m_2 & & & \\ & & \ddots & \ddots & \\ & & \ddots & m_{n-1} & \\ & & & & m_n \end{bmatrix} \tag{25}$$

In this example equal stiffnesses and equal masses are assumed as shown in the following:

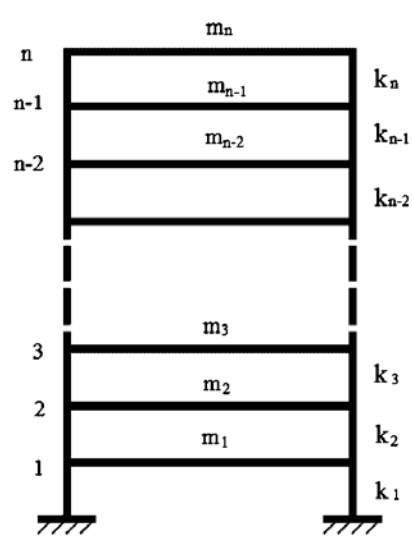


Fig. 5 A shear frame

$$k_i = k \quad , m_i = m \tag{26}$$

For calculating natural circular frequencies and mode shapes of structure, consider $\det(\mathbf{K} - \mathbf{M}\omega^2) = 0$. Consider equal stiffnesses and equal masses for the stories, then the eigenvalues and eigenvectors for these matrices will be as:

$$\lambda_k = b + 2\sqrt{ac}\ Cos[\frac{2k\pi}{2n+1}], \quad k = 1,2,...,n, \tag{27}$$

$$u_j^{(k)} = \rho^{j-1}\ Sin[\frac{2k\ j\pi}{2n+1}], \quad j = 1,2,...,n, \tag{28}$$

In this case, the natural circular frequencies of the structure are as the following:

$$\lambda_i = k(2 + 2\ Cos[\frac{2i\pi}{2n+1}]), \quad i = 1,2,...,n, \tag{29}$$

$$\omega = \sqrt{\frac{\lambda_i}{m}} = 2\,Cos[\frac{i\pi}{2n+1}]\sqrt{\frac{k}{m}} \tag{30}$$

For ordering natural circular frequencies from small to large, the Eq. (30) is rewritten as:

$$\omega_i = 2\,Cos[\frac{(n-i+1)\pi}{2n+1}]\sqrt{\frac{k}{m}} \tag{31}$$

And the corresponding mode shapes are given as:

$$\varphi_{ij} = Sin[\frac{j(2i-1)\pi}{2n+1}]\frac{2}{\sqrt{m(2n+1)}} \quad , i = 1,...,n \tag{32}$$

Considering that the circular frequencies and mode shapes are obtained as a closed form solution, now by the presented method, the response of structure subjected to arbitrary load can be obtained as a closed form solution. At first generalized force for the *i*th mode is obtained in the following:

$$p_i = \{Sin[\frac{(2i-1)\pi}{2n+1}], Sin[\frac{2(2i-1)\pi}{2n+1}], ..., Sin[\frac{n(2i-1)\pi}{2n+1}]\}\begin{Bmatrix}1\\1\\\vdots\\1\end{Bmatrix}\frac{2p(t)}{\sqrt{m(2n+1)}} \tag{33}$$

After simplification of Eq. (33), we obtain:

$$p_i = -Tan[\frac{(n+i)\pi}{2n+1}]\frac{2p(t)}{\sqrt{m(2n+1)}} \tag{34}$$

Then the magnitude of p_k, which is used in the closed form solution for response of the structure is obtained as:

$$p_k = -Tan[\frac{(n+i)\pi}{2n+1}]\frac{2h^2 p(t)}{\sqrt{m(2n+1)}} \tag{35}$$

Where h is the number of time steps. Using Eq. (36), the response of each story of structure can be calculated.

$$u_c = \sum_{i=1}^{n} \varphi_{ij} \; y_c \tag{36}$$

As an example, a twenty-storey structure is considered. The stiffness of each storey is taken as $k = 200 \times 10^3 \, KN/m$ and the mass of each storey is $m = 60 \, Mg$. Time history function of the load imposed on each storey is shown in Fig. 6.

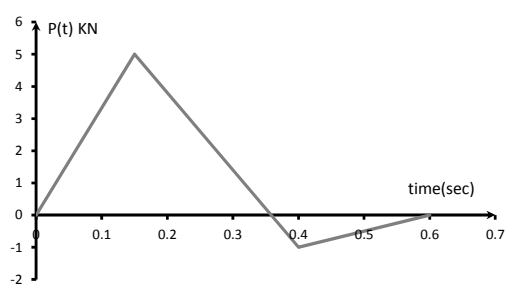


Fig. 6 The loading considered in Example 2

Using the presented method, it is very easy to calculate the natural circular frequencies and made shapes. Also considering Eq. (35), the generalized force for the *i*th mode can be obtained. Natural circular frequencies and damping ratio for four modes are shown in Table 2. Finally the responses of the first four modes of the structure for 2.5 second are obtained and shown in Fig. 7.

Table 2: Natural periods of the structure of Example 2

Mode number	1	2	3	4
ω_i	4.42	13.24	21.98	30.59
ξ_i	0.05	0.05	0.069	0.092

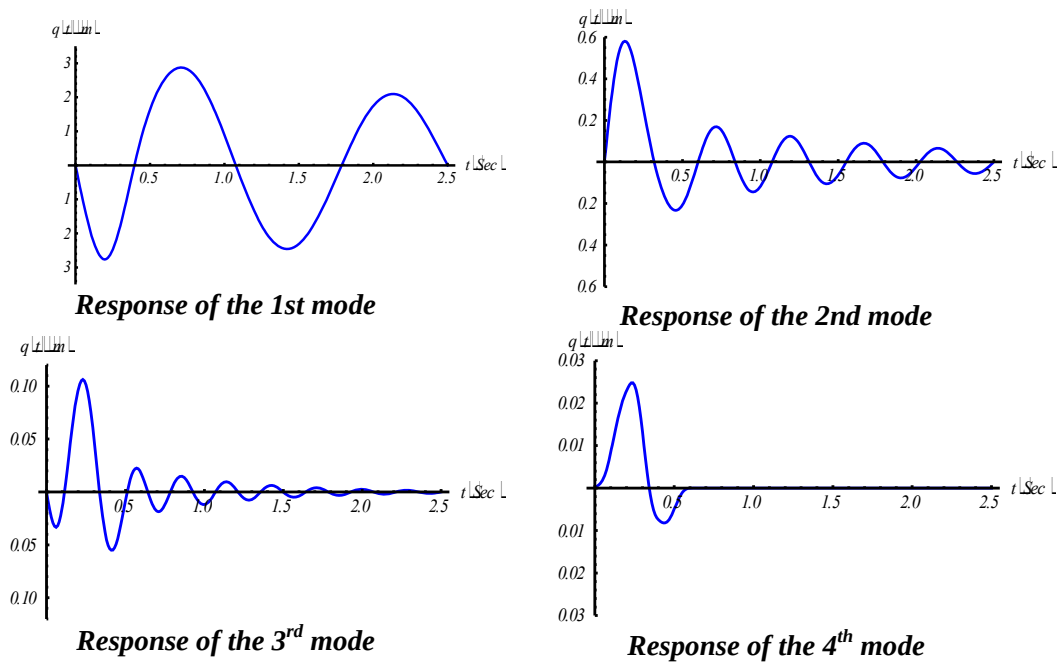


Fig. 7 Response of the first 4 modes of the structure

Example 3: In this example a 7-story plane steel frame is considered as shown in Fig. 8. The lump mass of each storey is 85.5 kN.s^2/m. Cross section of all columns is taken as $W14$ and all the beams are $W24$, Ref. [13]. The frame is subjected to El-Centro record. The response of structure is obtained by the present.

First the natural circular frequencies and corresponding mode shapes are obtained by $\det(\mathbf{K}-\mathbf{M}\omega^2)=0$. The first four modes of frame are shown in Table 3.

Table 3: Natural periods of the first 4 modes of Example ٣

Mode number	1	2	3	4
ω_i	4.947	14.612	26.179	39.26

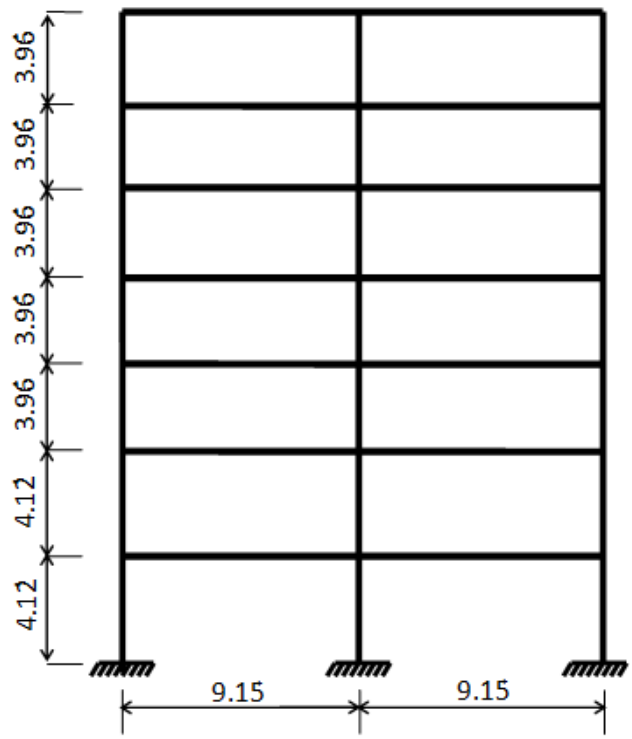
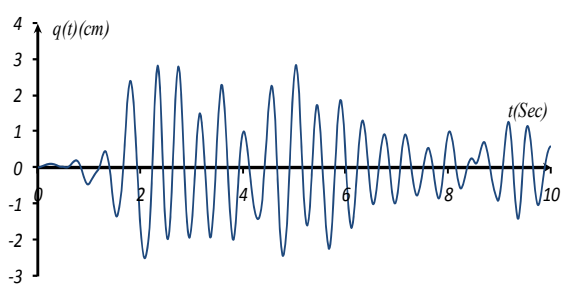


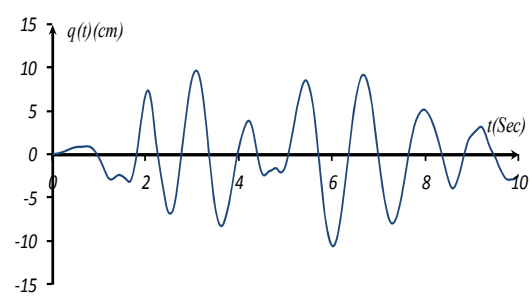
Fig. 8 A 7-story planar frame

Response of each mode is then obtained by Eq. (20). These responses for the first four modes are illustrated in Fig. 9. Then story displacement can be obtained by Eq. (36).

It should be mentioned that if mass and stiffness of each storey is assumed like the second example, then a closed form solution for response of each storey can be obtained. Using the present method the nonlinear seismic analysis of structures can be performed.



Response of the 2nd mode



Response of the 1st mode

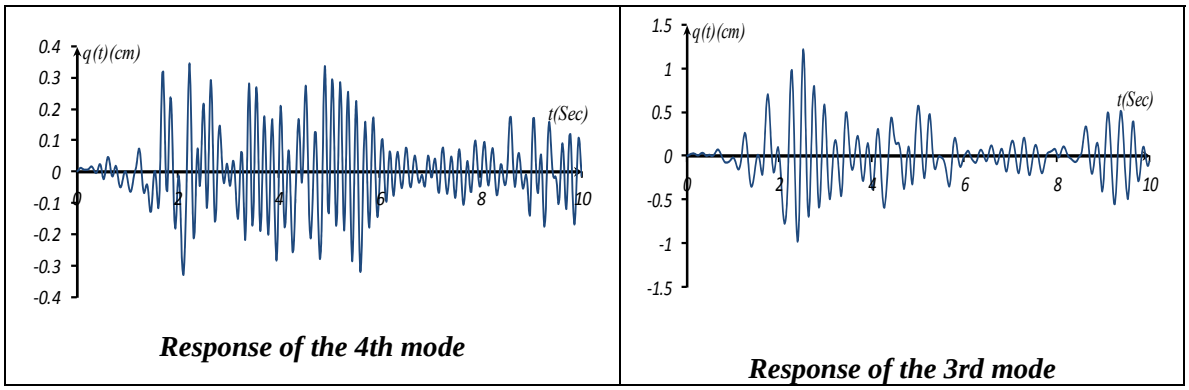


Fig. 9 Response of the first 4 modes of the structure in Example 3

4. CONCLUDING REMARKS

As mentioned before for dynamic and nonlinear analysis of structures subjected to arbitrary loading using of numerical method are necessary. In this paper a method based on finite difference method is presented for calculating the eigenvalues and eigenvectors of tri-diagonal matrices. Then closed form solutions are presented for dynamic response of undamped and damped structures subjected to arbitrary time history load, which can calculate the exactly response in any instance. Using this method, nonlinear analysis of structures can easily be performed. By this method formation of the response spectrum is very easy and can be more efficient than the usual method. Since by this method the response can be calculated in every time interval independently, it can have many applications in controlling structures problems. The important point about the presented method is that the response of structure does not depend on the responses in the previous instances; in the other hand, this method is not a step-by-step approach, which is not true for other numerical algorithm. For some other closed form solutions of symmetric and regular structures the interested reader may refer to Kaveh [14].

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