NEWMARK IMPLICIT TIME INTEGRAL FOR APPLICATION IN PSEUDO-DYNAMIC TESTING – MATHEMATICAL VERIFICATION

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ABSTRACT

In recent years, Pseudo-dynamic (PsD) technique is being adopted as an alternate to conventional shake-table technique to evaluate the seismic performance of structures. The shake-table technique has the merit of simulating all the three force parameters namely inertial, damping and elastic forces in the tested structure realistically; however the technique needs sophisticated shake-table driven by servo controlled actuators with appropriate control electronics. On contrary, PsD technique simulates the three force parameters by using a static actuator through application of an equivalent pseudo-dynamic force system with computation of inertial forces in the back-ground. Such a hybrid technique needs specialized algorithm based on an appropriate mathematical model for the off-line time integration and computation of inertial forces. Several time integrals have been proposed for application in PsD testing and majority of them are derived from Newmark-β family of algorithms. The traditional PsD testing uses constant acceleration version of Newmark time integral in explicit form for mathematical simplicity. This simplified explicit formulation results in numerical damping leading to considerable amplitude error in PsD testing, limiting its application to simple structures. However, for complicated structures improvement is needed in the time integral form leading to unconditional stability and zero numerical damping. This paper presents an improved form of Newmark implicit time integral for PsD testing. The improvement is based on the inclusion of an additional term in displacement predictor, which not only renders the algorithm more consistent, but also eliminates numerical damping and makes the algorithm unconditionally stable. The paper presents the analytical study carried out on the stability and energy dissipation properties of the improved time integral by evaluating its spectral characteristics for verifying its

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suitability in PsD testing.

1. INTRODUCTION

Earthquakes are potentially devastating natural events which threaten lives, destroy property, and disrupt life-sustaining services and societal functions. Indian peninsula has a very real earthquake problem and the past two decades have seen devastating earthquakes striking India with frightening regularity. India's four recently upgraded seismic zones as per the Indian Standard Code of Practice (IS 1893(2002): Part 1) [1] also emphasize that 59% of the land area in India is under moderate to severe earthquake hazard that assumes criticality in the context of all scales of the built environment. There are approximately 12 crore buildings in seismic zones III, IV and V. Most of these buildings are not earthquake resistant and are potentially vulnerable to collapse in the event of a high intensity earthquake. Hence evaluating the adequacy of seismic performance of civil engineering structures has come into focus following the damage and collapse of numerous structures during recent earthquakes. In addition, the adequacy of seismic performance of the older structures in regions of high seismicity, which were designed prior to the advent of revised seismic deign codes, is also a matter of growing concern.

1.1 Seismic performance evaluation methods

Several experimental methods *(Kausel 1998)*[2] are used to simulate and evaluate the seismic performance of structures and structural systems. These include, quasi-static testing, effective force testing and shake-table testing. Among them, the shake-table technique has the merit of simulating all the three force parameters namely inertial, damping and elastic forces in the tested structure realistically; however the technique needs sophisticated shake-table driven by servo controlled actuators with excellent control electronics. In the absence of such an expensive shake-table facility, it is possible to simulate the three force parameters using a static actuator through application of an equivalent pseudo-dynamic force system by computation of inertial forces in the back-ground. For such a hybrid Pseudo-dynamic (PsD) method, a specialized algorithm based on an appropriate mathematical model *(Takanashi et al., 1975)* [3] is needed for the off-line time integration and computation of inertial forces such that the forces are applied statically through static actuators. Restoring forces offered by the structure is experimentally measured on-line at each time step and reflects the actual in-elastic and energy dissipation characteristics of the structure.

1.2 Pseudo-dynamic testing

The PsD testing resembles the quasi-static method in that it also consists in applying slowly varying forces to the test structure (Yamazakhi et al. [4]; Mahin et al. [5]; Nakashima et al. [6]). However, during testing, the motions and deformations observed in the test structure are used to infer the inertial forces that the structure would have been exposed to the actual

earthquake; this information is then fed back into a control engine so as to determine and adjust the effective dynamic displacements/forces that must be applied onto the structure (Shing and Mahin [7]). These pseudo-dynamic forces are typically accomplished by means of actuators pushing against a large reaction wall. This alternate seismic performance evaluation methodology is picking up in the recent years and it is essential for to-days needs of growing India with enhanced seismic risk. This method has the advantage of testing large and tall test structures with center of mass well above the base which are normally can not be tested on a shake table for evaluating their seismic performance. As this method involves application of dynamic forces in an equivalent static mean through static actuators, close monitoring of the structural behavior including crack initiation, crack growth and stiffness degradation is also becomes possible. The draw back in such a hybrid method is the lack of simulation of strain rate effects which may not be critical under seismic loads. Also the method is time consuming due to its iterative nature.

2. MATERIALS AND METHODS

Pseudo-dynamic (PsD) testing is a combined computational and experimental technique for evaluating dynamic systems originally proposed by Takanashi et al. [3]. The method relies on modeling inertial and damping forces computationally, while the nonlinear restoring forces are measured experimentally. Dynamic equilibrium equations can generally be expressed as

$$M \frac{d^2 x}{dt^2} + C \frac{dx}{dt} + r(x) = f \tag{1}$$

Where, M and C are mass and viscous damping matrices and x, r and f are the displacement, restoring force and applied force vectors respectively. It is assumed that $r(x)$ is the only source of nonlinearity which can be obtained accurate enough through experimental measurements. The PsD test method uniquely utilizes both computational and experimental terms to form the equation of motion (Equation 1). The response is obtained by discretising time and calculating it in a step-by-step manner.

A time stepping formulation computes a displacement step which is subsequently imposed on the structure by means of computer controlled servo-hydraulic actuators as shown in Figure 1. Once the structure has been deformed, the resulting restoring forces are measured. Based on these restoring forces and the current damping and applied forces, the resulting new acceleration may be calculated. A new displacement step can then be calculated, and the next step has thus commenced. In comparison to shaking table testing, there are some important differences. As the PsD testing is carried out in a step-by-step fashion, it is clear that it is unrealistic to be able to progress the test in real time. Furthermore, as inertial effects are modeled computationally, such forces need not and should not exist in the physical model. The time scale of a typical test is therefore expanded in magnitude which has both beneficial and adverse effects. The fact that the structure is displaced slowly (and can even be stopped) provides a good opportunity for inspection and

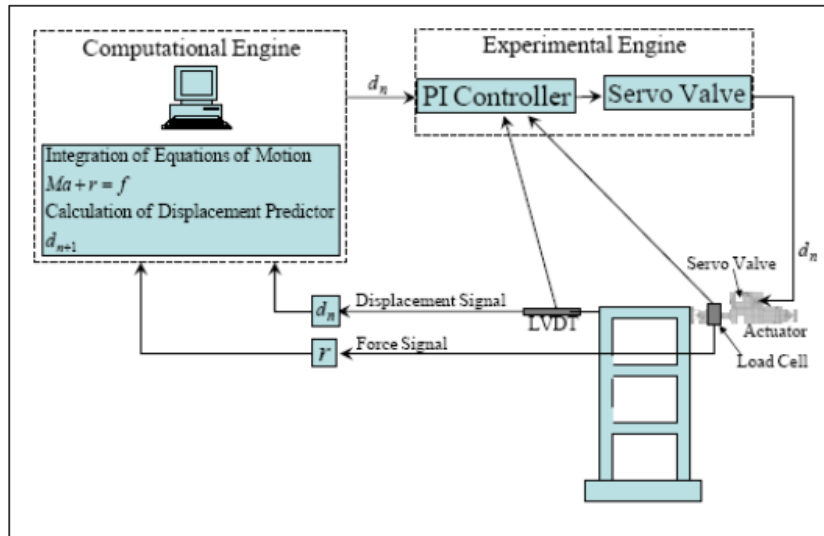any detailed readings to be taken; however, the strain rate effects on material response are neglected.

Figure 1. Computational and experimental components of a typical pseudo-dynamic (PsD) test set-up

2.1 Mathematical formulation of PsD testing

Several time stepping formulation have been proposed for application in PsD testing (Buonopane [10]; Bursi and Shing [11]; Combescure and Pegon [12]; Chang [13]). The majority of these formulations are explicit due to the fact that the nonlinear structural restoring forces at the end of any time step are unknown and displacement iterations in PsD test are undesirable as these might result in partial unloading (Shing and Manivannan [14]). Although implicit methods have the advantage of being unconditionally stable, the duration of the time steps still has to be limited for accuracy purposes due to rapid changes in both loading and stiffness.

2.2 PsD formulation based on Newmark explicit time integral

The integral form of the Newmark explicit was initially proposed by Chang et al. [15] by integrating the equation of motion (Equation 1) in its incremental form once with respect to time. It is argued that such a form is better suited for rapidly varying excitation force and rapidly varying level of nonlinearity of the restoring force. The improved accuracy originates from the fact that by performing the integration, the equilibrium is satisfied over the time step as a whole rather than at its start or at its end. The solution involves utilization of the time integral of the force for each time step which can be found reasonably accurately through some simple numerical integration and sub-stepping. On the other hand, linearization of the time-force integral, when sampling the excitation accelerogram at large time step intervals, may lead to significant inaccuracies. Similarly, for the restoring force, a

linearised relationship between the start and the end values may very poorly represent the actual variation of the force over the time step. By linear approximation between the start and the end points, it underestimates the restoring force during loading and overestimates it during unloading – this evidently adds energy to the system and could potentially lead to instability. While the error in the linearization of the excitation force is of a random nature, it should be noted that the error associated with restoring force is systematic and cumulative. The applicability and the advantages of implementing the integral form method into PsD testing have been thoroughly investigated in Chang et al. [15]. The basic Newmark implicit relations and integrating Equation 1 yields

$$M\Delta\frac{dx}{dt} + C\Delta x + \Delta\int r(x)dt = \Delta\int fdt \tag{2a}$$

$$d_{n+1} = d_n + \Delta t v_n + \left(\frac{1}{2} - \beta\right)(\Delta t)^2 a_n + \beta(\Delta t)^2 a_{n+1} \tag{2b}$$

$$v_{n+1} = v_n + (1-\gamma)\Delta t a_n + \gamma\Delta t a_{n+1} \tag{2c}$$

Where, Δt is the time step duration, d and v the displacement and velocity, respectively, and Δ indicates the change over one time step. The Chang formulation builds on integrating the incremental equations of the explicit format of the Newmark method by using $\beta = 0$ in the Newmark implicit relations. Then the basic Newmark explicit equations are

$$M\Delta a_{n+1} + C\Delta v_{n+1} + \Delta r_{n+1} = \Delta f_{n+1} \tag{3a}$$

$$d_{n+1} = d_n + \Delta t v_n + \frac{1}{2}(\Delta t)^2 a_n \tag{3b}$$

$$v_{n+1} = v_n + \frac{1}{2}\Delta t(a_n + a_{n+1}) \tag{3c}$$

Equations 3 are integrated once again with respect to time which leads to the following equations:

$$M\Delta v_{n+1} + C\Delta d_{n+1} + \Delta\int r_{n+1}dt = \Delta\int f_{n+1}dt \tag{4a}$$

$$\int d_{n+1}dt = \int d_n dt + \Delta t d_n + \frac{1}{2}(\Delta t)^2 v_n \tag{4b}$$

$$d_{n+1} = d_n + \frac{1}{2}\Delta t(v_n + v_{n+1}) \tag{4c}$$

In the usual Newmark explicit formulation (Equation 3), the equations are solved for the change in acceleration, and the equations of motion in the integral form, Equation 4, are now solved for the change in velocity. Additionally, the integral form has an expression for the time-integral of displacement instead of the displacement predictor Δd_{n+1} in the usual form. More importantly, the term in the integral form no longer represents an explicit prediction

that may be used as an initial displacement step in pseudo-dynamics. The displacement step is now an implicit function of v_n and v_{n+1}, and can be found from Equation 4c, which in turn requires the solution of Equation 4a to obtain the velocity at the end of the time step v_{n+1}. In effect, the action of integrating the set of equations has rendered the method implicit in the sense that the predictor displacement cannot be deduced directly any more. The integral form formulation also requires an assessment of the integral of the restoring force before the displacement predictor can be calculated. Such an estimate enables the solution for Δv_{n+1}, to be found, which in turn produces an explicit predictor for the displacement Equation 4(c) which is needed for the pseudo-dynamic implementation. The restoring force and its time integral are nonlinear functions of displacement, and can no longer be obtained directly, as no predictor displacement step exists to be imposed. In order to be able to utilize the formulation, Chang et al. [15]suggest multiplying Equation 4b by the tangent stiffness, and an explicit expression of the integral of the restoring force at $t = t_{n+1}$ may be found (here expressed for an SDOF system) as outlined in Equation 5.

$$\int r_{n+1} dt = \int r_n dt + \Delta t k d_n + \frac{k}{2}(\Delta t)^2 v_n = \int r_n dt + \Delta t r_n + \frac{k}{2}(\Delta t)^2 v_n \qquad (5)$$

The physical interpretation of the above expression can be seen from the graph of restoring force vs. time as shown Figure 2, where $\int r_n dt$ indicates the restoring force time area at a given time t, while the sum of the two remaining terms represents the projected trapezoidal area assuming that a constant velocity exists until the end of the step.
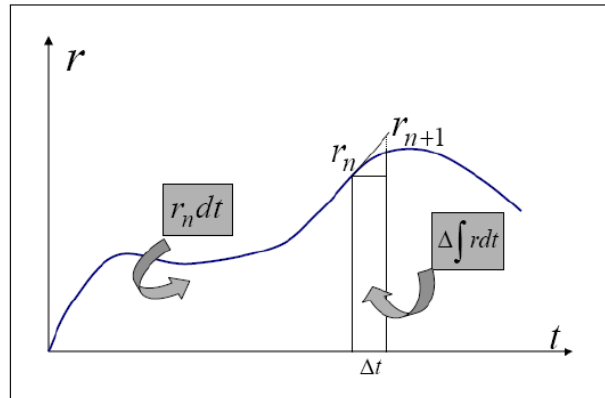

Figure 2. Approximation of restoring force $\Delta r_{n+1} dt$

Such a procedure tentatively assumes that the tangent stiffness is known, or may be obtained somehow, which will normally not be the case in PsD testing. Only for the SDOF and for certain simple MDOF structures, may the stiffness matrix be computed from experimental data, and then, only once a time step has been completed. To overcome the problem of the unknown tangent stiffness matrix, Chang et al. [15] suggest replacing it with the initial stiffness term. The error involved is not large as the tangent stiffness is required

only in the second order term on the right hand side of Equation 5. In any case, once an expression for the integral of the restoring force exists, Chang et al. [15] suggest a solution procedure where they solve for Δv_{n+1} from Equation 4a by substituting Equation 4c for d_{n+1}. When considering an SDOF system, the velocity change can be expressed as

$$\Delta v_{n+1} = \left(m + \frac{\Delta t}{2} c \right)^{-1} \left(\Delta \int f_{n+1} dt - c\Delta t v_n - \Delta t r_n - \frac{k_0}{2} (\Delta t)^2 v_n \right) \tag{6}$$

Where, k_o is the initial stiffness in place of the tangent stiffness term. By substituting this result back into Equation 4c, a prediction for the change in displacement can finally be expressed as follows:

$$\Delta d_{n+1} = \Delta t v_n + \frac{\Delta t}{2} \left(m + \frac{\Delta t}{2} c \right)^{-1} \left(\Delta \int f_{n+1} dt - c\Delta t v_n - \Delta t r_n - \frac{k_0}{2} (\Delta t)^2 v_n \right) \tag{7}$$

Which, when added on to the previous displacement value, furnishes an explicit displacement predictor to be applied in pseudo-dynamic tests similarly to any other displacement predictor. However, the method now departs from the traditional procedure in PsD testing. While the predictor displacement step is being imposed on the structure, the induced restoring force is continuously measured, and its corresponding time integral is evaluated numerically. At the end of the step, this will represent an experimentally evaluated change in the time integral of the restoring force, the same term as the one which was earlier estimated in Equation 5. In general, due to the material nonlinearity, the restoring force will not follow the linear extrapolation as estimated, and the change in the time integral of this restoring force will in reality be smaller than estimated. For this reason, Chang's algorithm then recalculates Δv_{n+1} based on the measured time integral of the restoring force. At this stage, Δd_{n+1} may or may not be recalculated based on the updated Δv_{n+1} in Equation 4c. Irrespective of whether the displacement increment Δd_{n+1} is then recalculated, there is a numerical damping present which may be negative or positive depending on which reference values for the restoring force are used at the beginning of the displacement increment. The effect will be present irrespective of whether the tangential or initial stiffness term is used in Equation 5.

2.3 PsD formulation based on Newmark implicit time integral
There is an inconsistency in the integral form of the Newmark explicit formulation when recalculating Δv_{n+1} based on the updated $\Delta r_{n+1} dt$ (Algaard et al. [16]). Δv_{n+1} indeed has to be recalculated; otherwise, the information about the experimentally measured restoring forces is never taken into account. The predictor Δd_{n+1} is based on an estimation of Δv_{n+1} which in turn build on an approximation of the restoring force as expressed in Equation 5. Once the integral of the restoring force has been obtained, the integrated equation of motion, Equation 4a, can be applied. Expanding Equation 4c yields

$$d_{n+1} = d_n + \frac{1}{2}\Delta t(v_n + v_{n+1}) = d_n + \frac{1}{2}\Delta t(v_n + v_n + \Delta v_{n+1})$$

$$........ = d_n + \Delta t v_n + \frac{1}{2}\Delta t \Delta v_{n+1}$$

(8)

Furthermore, Equation 4a assumes the integral of the restoring force over that time step, $\Delta \int r_{n+1}dt$, to be determined by computing the time integral of the restoring force over the time step, as indicated in Figure 2. Assuming linear stiffness for simplicity, the exact expression for $\Delta \int r_{n+1}dt$ will be as follows:

$$\Delta \int r_{n+1}dt = \int r_{n+1}dt - \int r_n dt = \Delta tk\left(\frac{d_n + d_{n+1}}{2}\right)$$

(9)

Equation 8 can now be substituted for d_{n+1} in Equation 9 to yield the following:

$$\Delta \int r_{n+1}dt = \Delta tk\left(\frac{d_n + d_n + \Delta t v_n + 1/2\Delta t \Delta v_{n+1}}{2}\right)$$

(10)

This can be further manipulated into

$$\Delta \int r_{n+1}dt = \Delta tk d_n + \frac{1}{2}\Delta t^2 k v_n + \frac{1}{4}\Delta t^2 k \Delta v_{n+1}$$

(11)

By comparing Equation 11 with Equation 5, it is clear that Equation 11 now contains one additional term, $(\Delta t^2 k \Delta v_{n+1})/4$. This term corresponds to the term which is omitted in the standard implicit Newmark algorithm to render it explicit (i.e., $\beta = 0$), and the omission of this term is the cause of the numerical damping invariably present in the integral form of the algorithm. However, omitting the equivalent term in the integral form of the method does not render the method explicit; in fact, its omission has no bearing on the nature of the algorithm.

The integral form of the Newmark algorithm has been made explicit through an estimation of the time integral of the restoring force, which enables the calculation of Δv_{n+1} followed by Δd_{n+1}. At this point, it becomes clear that there is no reason why the seemingly implicit additional term in Equation 11 cannot be included in the estimation of the time integral of restoring force, as the implicit variable is the actual unknown, the expression is trying to represent. The situation is clarified through the following argument, where Equation 11 has been substituted into Equation 4a and solved for Δv_{n+1} to yield an alternative expression for Equation 6.

$$\Delta v_{n+1} = \left(m + \frac{\Delta t}{2}c\right)^{-1}\left(\Delta \int f_{n+1}dt - c\Delta t v_n - \Delta t r_n - \frac{k_0}{2}(\Delta t)^2 v_n - \frac{1}{4}\Delta t^2 k_0 \Delta v_{n+1}\right) \qquad (12)$$

The unknown, Δv_{n+1}, is present on both sides of the equation, but through further rearrangement

$$\Delta v_{n+1} = \left(m + \frac{\Delta t}{2}c\right)^{-1}\frac{1}{4}\Delta t^2 k_0 \Delta v_{n+1}$$

$$.......... = \left(m + \frac{\Delta t}{2}c\right)^{-1}\left(\Delta \int f_{n+1}dt - c\Delta t v_n - \Delta t r_n - \frac{k_0}{2}(\Delta t)^2 v_n\right)\left(1 + \left(m + \frac{\Delta t}{2}c\right)^{-1}\frac{1}{4}\Delta t^2 k_0\right)\Delta v_{n+1}$$

$$.......... = \left(m + \frac{\Delta t}{2}c\right)^{-1}\left(\Delta \int f_{n+1}dt - c\Delta t v_n - \Delta t r_n - \frac{k_0}{2}(\Delta t)^2 v_n\right)$$

an explicit expression for the velocity can finally be found.

$$\Delta v_{n+1} = \frac{\left(m + \frac{\Delta t}{2}c\right)^{-1}\left(\Delta \int f_{n+1}dt - c\Delta t v_n - \Delta t r_n - \frac{k_0}{2}(\Delta t)^2 v_n\right)}{1 + \left(m + \frac{\Delta t}{2}c\right)^{-1}\frac{1}{4}\Delta t^2 k_o} \qquad (13)$$

This equation may now be substituted directly into Equation 8, and rearranged to obtain a new explicit expression for the displacement predictor Δd_{n+1}:

$$\Delta d_{n+1} = \Delta t v_n + \frac{\frac{\Delta t}{2}\left(m + \frac{\Delta t}{2}c\right)^{-1}\left(\Delta \int f_{n+1}dt - c\Delta t v_n - \Delta t r_n - \frac{k_0}{2}(\Delta t)^2 v_n\right)}{1 + \left(m + \frac{\Delta t}{2}c\right)^{-1}\frac{1}{4}\Delta t^2 k_0} \qquad (14)$$

By using Equation 14 rather than Equation 7 as the displacement predictor, the time stepping formulation has become implicit. To be exact, the formulation still requires a value of the tangent stiffness (k_0 terms in Equation 14), however this is also the case with the earlier explicit version of the formulation and other implicit formulations (Combescure and Pegon [12]). The same term is now merely present in one additional place, and as with the explicit version, the initial stiffness or an estimate of the tangent stiffness can be used. Principal differences between the two PsD formulations are summarized in Table 1.

Table 1: Comparison of chang formulation based on newmark explicit time integral and improved formulation based on newmark implicit time integral

Chang formulation based on Newmark explicit time integral	
Predictor d_{n+1}	$\Delta t v_n + \dfrac{\Delta t}{2}\left(m + \dfrac{\Delta t}{2}c\right)^{-1}\left(\Delta \int f_{n+1}dt - c\Delta t v_n - \Delta t r_n - \dfrac{k_0}{2}(\Delta t)^2 v_n\right)$
$\int d_{n+1}dt$	$\int d_n dt + \Delta t d_n + \dfrac{1}{2}(\Delta t)^2 v_n$
Improved formulation based on Newmark implicit time integral	
Predictor d_{n+1}	$\Delta t v_n + \dfrac{\dfrac{\Delta t}{2}\left(m + \dfrac{\Delta t}{2}c\right)^{-1}\left(\Delta \int f_{n+1}dt - c\Delta t v_n - \Delta t r_n - \dfrac{k_0}{2}(\Delta t)^2 v_n\right)}{1 + \left(m + \dfrac{\Delta t}{2}c\right)^{-1}\dfrac{1}{4}\Delta t^2 k_0}$
$\int d_{n+1}dt$	$\int d_n dt + \Delta t d_n + \dfrac{1}{2}(\Delta t)^2 v_n + \dfrac{1}{4}(\Delta t)^2 \Delta v_{n+1}$

3. MATHEMATICAL VERIFICATION OF THE IMPROVED PSD FORMULATION

The suitability of the proposed formulation based on Newmark implicit relations is verified through evaluating its stability and numerical damping properties. The stability properties of a time integral formulation are studied by considering the spectral properties of its recursive amplification matrix. Considering the state vector of the system at time $t = t_n$, the integral form of the numerical time integral form yields

$$X_n = \begin{bmatrix} \int d_n dt \\ \Delta t d_n \\ \Delta t^2 v_n \end{bmatrix} \tag{15}$$

For stability purposes, one can ignore the external load vector and damping forces, thus the recurrent relationship between the state vector at $t = t_n$ and $t = t_{n+1}$ can be expressed as (Bathe and Wilson [17]).

$$X_{n+1} = [A]X_n \tag{16}$$

Where, $[A]$ is the recursive amplification matrix.

3.1 Stability and dissipation properties of PsD formulation based Newmark explicit relations

Considering first the explicit time integral form, Equations 4 can be expressed in terms of the variables of the state vector as

$$\int d_{n+1}dt = \int d_n dt + \Delta t d_n + \frac{1}{2}(\Delta t)^2 v_n$$

$$d_{n+1} = d_n + \frac{1}{2}\Delta t(v_n + v_{n+1})$$

$$\Delta v_{n+1} = M^{-1}\left[- \int_{n}^{n+1} r_{n+1}dt \right]$$

(17)

Depending on the precise implementation of the time integral form, i.e., whether d_{n+1} is updated following the recalculation of v_{n+1} or not, the exact expressions for d_{n+1} and $r_{n+1}dt$ will differ.

Assuming initially that d_{n+1} is not recalculated, the predicted d_{n+1} remains, and the term $r_{n+1}dt$ will be a function of the restoring force both at the start and at the end of the predicted step yielding the second of Equations 18. d_{n+1} will thus no longer be represented by the implicit expression in Equation 17, rather by a simplification of Equation 7 containing only the terms relevant for stability analyses. $r_{n+1}dt$ may be defined in terms of d_{n+1} or by the expression for the prediction step; however, this will in the end lead to the same amplification matrix. Assuming linear stiffness for simplicity, $r_{n+1}dt$ may be expressed as $\frac{\Delta tk}{2}(d_n + d_{n+1})$, yielding the third equation of Equations 18.

$$\int r_{n+1}dt = \int r_n dt + \Delta t d_n + \frac{1}{2}(\Delta t)^2 v_n$$

$$d_{n+1} = d_n + \Delta t v_n + \frac{1}{2}\Delta t M^{-1}\left(- \Delta t k d_n - \frac{k}{2}(\Delta t)^2 v_n \right)$$

(18)

$$v_{n+1} = v_n + M^{-1}\left(- \frac{\Delta tk}{2}(d_n + d_{n+1}) \right)$$

Multiplying the second and third equations by Δt and Δt^2 respectively and substituting Ω^2 and $\Delta t^2 k/m$ yield Equations 19.

$$\int r_{n+1}dt = \int r_n dt + \Delta t d_n + \frac{1}{2}(\Delta t)^2 v_n$$

$$\Delta t d_{n+1} = \Delta t d_n + \Delta t^2 v_n + \frac{1}{2}\Omega^2 t d_n - \frac{1}{4}\Omega^2(\Delta t)^2 v_n$$

(19)

$$\Delta t^2 v_{n+1} = \Delta t^2 v_n - \Omega^2 \frac{\Delta t d_n}{2} - \Omega^2 \frac{\Delta t d_{n+1}}{2}$$

After sorting terms at $t = t_{n+1}$ and $t = t_n$ and expressing them in matrix form (Geradin and Rixen [18]), the amplification matrix is obtained as

$$A = \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 - \frac{\Omega^2}{2} & 1 - \frac{\Omega^2}{4} \\ 0 & -\Omega^2 + \frac{\Omega^4}{4} & 1 - \frac{\Omega^2}{2} + \frac{\Omega^4}{8} \end{bmatrix} \tag{20}$$

Which clearly differs from the normal newmark explicit matrix, Shing and Mahin [19]. Stability of a time integral formulation is ensured when the spectral radius of the amplification matrix does not exceed unity, Golley and Amer [20]. In the above matrix, one eigen value will be equal to unity, while the other two will form a pair of complex conjugates. Corresponding moduli have been plotted below as a function of Ω in Figure 3.
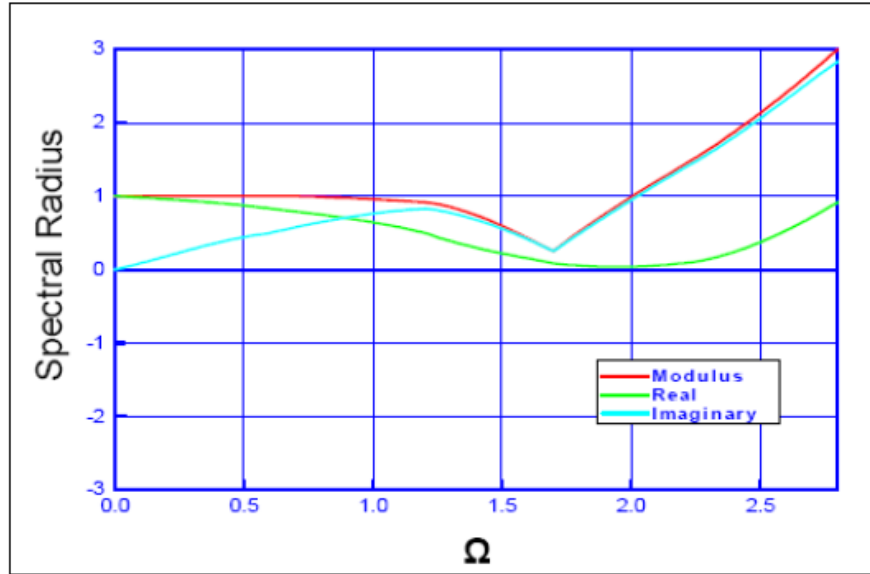

Figure 3. Plot of spectral radius variation along Ω for newmark explicit time integral form
(Chang et al. [15])

From Figure 3, both the expected stability limit of 2.0 and the existence of noticeable numerical damping can be seen for the values of Ω exceeding 0.5. If however the d_{n+1} is recalculated once the corrected v_{n+1} has been found, the situation is somewhat different.

Equations 18 will be altered such that the displacement predictor is no longer present in the definition of d_{n+1}, but remains in the expression for v_{n+1}. This yields Equations 21, which can be represented by the amplification matrix shown in Equation 22, obtained the same way as Equation 20.

$$\int r_{n+1} dt = \int r_n dt + \Delta t d_n + \frac{1}{2}(\Delta t)^2 v_n$$

$$d_{n+1} = d_n + \frac{1}{2}\Delta t v_n + \frac{1}{2}\Delta t v_{n+1}$$

$$v_{n+1} = v_n + M^{-1} - \left(\frac{\Delta t k}{2}\left(2d_n + \Delta t v_n + \frac{1}{2}\Delta t M^{-1}\left(-\Delta t k d_n - \frac{k}{2}(\Delta t)^2 v_n\right)\right)\right)$$

(21)

$$A = \begin{bmatrix} 1 & 1 & \dfrac{1}{2} \\ 0 & 1 - \dfrac{\Omega^2}{2} + \dfrac{\Omega^4}{8} & 1 - \dfrac{\Omega^2}{4} + \dfrac{\Omega^4}{16} \\ 0 & -\Omega^2 + \dfrac{\Omega^4}{4} & 1 - \dfrac{\Omega^2}{2} + \dfrac{\Omega^4}{8} \end{bmatrix}$$

(22)

This time integral form exhibits similar stability and damping characteristics as the standard Newmark explicit; perfect energy conservation up to the stability limit of 2.0. However, as the algorithm stands, it cannot be directly implemented into a PsD test. This is because r_n is in fact unknown at the start of the time step. The reason for this is that d_n was recalculated after the completion of the previous step, and the restoring force caused by it is thus unknown. The correct procedure would require the recalculated d_{n+1} to be imposed separately, and the corresponding restoring force re-measured. Such a procedure would lead to a double step implementation, but employing iterations in a PsD algorithm that is still only conditionally stable seems inappropriate.

3.2 Stability and dissipation properties of the improved PsD formulation based on Newmark implicit relations
The effects of using the implicit version of the time integral form with the modified displacement predictor are substantial. Not only does the method avoid the numerical damping associated with the Newmark explicit – integral form algorithm, but owing to the fact that the algorithm is now genuinely implicit, it also becomes unconditionally stable. This was initially noted through numerical experiments, but can also be confirmed analytically. By considering the expression for the time integral of displacement, the displacement and the velocity and using a similar procedure as with Equation 19, it leads to the following equations:

$$\int d_{n+1} dt = \int d_n dt + \Delta t d_n + \frac{1}{2}(\Delta t)^2 v_n - \beta\frac{\Omega^2}{1+\Omega^2/4}\Delta t d_n - \frac{1}{2}\beta\frac{\Omega^2}{1+\Omega^2/4}(\Delta t)^2 v_n \quad (23)$$

$$\Delta t d_{n+1} = \Delta t d_n + \Delta t^2 v_n - \gamma \frac{\Omega^2}{1+\Omega^2/4} \Delta t d_n - \frac{1}{2}\gamma \frac{\Omega^2}{1+\Omega^2/4}(\Delta t)^2 v_n$$

$$\Omega^2 \int d_{n+1} dt + (\Delta t)^2 v_{n+1} = \Omega^2 \int d_n dt + (\Delta t)^2 v_n$$

Where, β and γ are the parameters normally present in the Newmark algorithms which typically take the values of 0.25 and 0.50 respectively. Expressing the above equations in a matrix form yields again the recursive amplification matrix of the integration operator, *Geradin and Rixen* [18]:

$$A = \begin{bmatrix} 1 & 1-\beta\dfrac{\Omega^2}{1+\Omega^2/4} & \dfrac{1}{2}-\dfrac{1}{2}\beta\dfrac{\Omega^2}{1+\Omega^2/4} \\ 0 & 1-\gamma\dfrac{\Omega^2}{1+\Omega^2/4} & 1-\dfrac{1}{2}\gamma\dfrac{\Omega^2}{1+\Omega^2/4} \\ 0 & -\Omega^2\left(1-\beta\dfrac{\Omega^2}{1+\Omega^2/4}\right) & 1-\Omega^2\left(\dfrac{1}{2}-\dfrac{1}{2}\beta\dfrac{\Omega^2}{1+\Omega^2/4}\right) \end{bmatrix} \tag{24}$$

The complex expression for the eigen values of $[A]$ can be simplified to

$$(1-\lambda)\left(\lambda^2 +\left\{\left(-2+\frac{1}{2}\Omega^2\right)\left(1-\frac{1}{4}\xi^2\right)\right\}\lambda +1+\frac{\Omega^2}{2}-\frac{\Omega^2\xi^2}{8}-\frac{1}{2}\xi^2\right)=0 \tag{25}$$

Where, λ are the eigen values and ξ^2 is expressed as

$$\xi^2 = \frac{\Omega^2}{1+\dfrac{1}{4}\Omega^2} \tag{26}$$

Eliminating $\lambda_1 = 1$ leaves the remaining second order equation:

$$\lambda_{2,3} = \frac{\left(-2+\dfrac{\xi^2}{2}+\dfrac{\Omega^2}{2}+\dfrac{\Omega^2\xi2}{8}\right)\pm\sqrt{-4\Omega^2+\dfrac{1}{4}\xi^4+\dfrac{1}{4}\Omega^4+\dfrac{3\xi^2\Omega^2}{2}-\dfrac{\Omega^4\xi^2}{8}-\dfrac{\Omega^2\xi^4}{8}+\dfrac{\Omega^4\xi^4}{64}}}{2} \tag{27}$$

By plotting the real and imaginary parts of the solution and computing the moduli, it can be seen that the moduli for $\lambda_{2,3}$ are also equal to unity for all Ω, as shown in Figure 4. This clearly yields the overall solution that $\rho(A) = 1$ for all Δt, which implies unconditional stability and perfect energy conservation without any numerical damping.
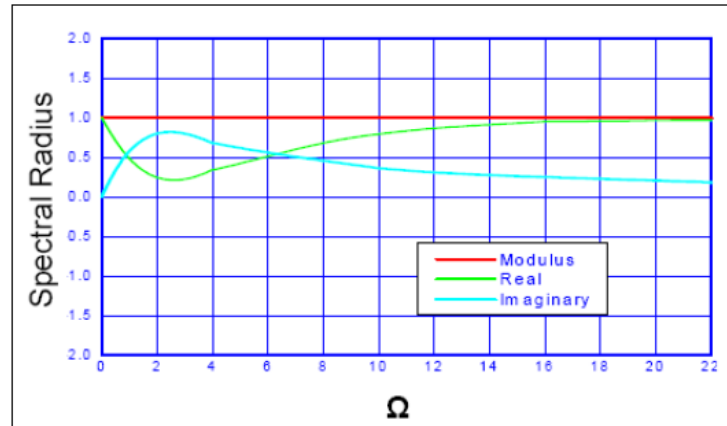
Figure 4. Plot of spectral radius variation along Ω for the proposed algorithm based on Newmark implicit time integral form

The modifications carried out on the Newmark explicit time integral form have shown to eliminate the numerical damping (or) amplitude error of the algorithm, and also improve the stability properties such that it is now unconditionally stable. Hence, the proposed algorithm (Sathish Kumar et al., [21]; Sathish Kumar et al. [22]) based on Newmark implicit time integral form found to be more appropriate for implementation in PsD testing.

4. SUMMARY AND CONCLUSIONS

The genesis, development and mathematical formulation of pseudo-dynamic (PsD) testing for experimental seismic performance evaluation of structures are presented in the paper in detail. The suitability of Newmark implicit time integral for PsD testing is verified numerically through studying its stability and numerical damping characteristics. The study showed that the Newmark implicit time integral is found to have unconditional stability and zero numerical damping leading to near-zero amplitude error in PsD testing for seismic performance evaluation of structures.

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