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A SECOND ORDER TIME INTEGRATION SCHEME FOR ELASTIC DYNAMIC PROBLEMS

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ABSTRACT

A time integration scheme is proposed for dynamic analysis of linear elastic problems. This method assumes higher order variation of the acceleration at each time step. Two variable parameters are used to increase the stability and accuracy of the method. In the proposed method, second order accuracy and unconditionally stable method is achieved for all values of the assumed parameters with and without numerical damping. Moreover, the proposed method controls numerical dissipation in the higher modes. Finally, the numerical results of the proposed method are compared with two classical methods; namely the average acceleration and the Wilson-θ methods.

Keywords: Time integration; stable scheme; numerical damping; structural dynamics.

1. INTRODUCTION

One of the procedures for calculating the dynamic response of structures is the direct numerical integration of the differential equation of motion. It can be used for both linear and nonlinear systems. Providing dynamic response at time t needs to be determined, the first step would be to subdivide the time interval between 0 and t into n time intervals (Δt_j). The time interval Δt_j between times t_j and t_{j+1} is usually taken to be of uniform duration Δt, therefore $\Delta t = t/n$. Then, it is assumed that the variation of acceleration, velocity, and displacement within each time interval Δt follows a special pattern (a polynomial of certain degree). Dynamic equilibrium is usually satisfied at those discrete times. Equations of step-by-step integration methods can be derived by using the Taylor series expansion. Classical methods such as the Newmark method [1] or the Wilson-θ method [2] assume a constant or linear variation for the variation of acceleration at each time step [3]. By increasing order of the variation of acceleration, higher accuracy is achieved as more terms are kept in the Taylor series expansion [4]. Another interesting technique is composite time integration. This method has a good

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efficiency in nonlinear dynamic problems [5-8]. In addition to the order of accuracy, stability, dissipation and dispersion errors are other significant factors for evaluating a time integration method. Numerical dissipation and dispersion are measured with the numerical damping ratio and the relative period error respectively [9].

In several time integration methods, the equation of motion is satisfied at the beginning of each time step in order to calculate the unknown values at the end of the step. These methods are called explicit methods. However, in implicit time integration methods, it is required to satisfy the equation of motion at the end of time step in order to calculate the unknown values at this point. A review of several implicit and explicit methods is found in [7, 10-16]. For large multi-degree of freedom systems, it is essential to apply unconditionally stable methods due to the fact that in conditionally stable methods, the time step Δt must be smaller than a critical time step, Δt_{cr} (proportional to the smallest natural period of the system). Consequently, it often involves using time steps that are much smaller than those needed for accuracy [9].

In this paper, a new implicit time integration scheme is introduced. In the proposed method, the acceleration varies in quadratic manner within each time step. Equations of presented method are developed from the Taylor series expansion. Considering those assumptions and employing the two parameters δ and α, a family of unconditionally stable schemes is obtained with high accuracy for solving the structural dynamic problems. Next, the requirements for unconditional stability of the new technique are presented. Finally, the accuracy of the proposed technique is evaluated.

2. PRESENT METHOD

In dynamics, the linear equation of motion is described as:

$$M\ddot{U}+C\dot{U}+KU=P \tag{1}$$

where M, C and K are mass, damping and stiffness matrices while P is the vector of applied forces; U , $\dot{U}$ and $\ddot{U}$ are the displacement, velocity and acceleration vectors, respectively. The symbols ${}^{t}U$, ${}^{t}\dot{U}$ and ${}^{t}\ddot{U}$ denote U , $\dot{U}$ and $\ddot{U}$ at time t. The Taylor series expansions of ${}^{t+\Delta t}U$ and ${}^{t+\Delta t}\dot{U}$ about time t are applied; as follows:

$${}^{t+\Delta t}U={}^{t}U+\Delta t\,{}^{t}\dot{U}+\frac{\Delta t^2}{2}\,{}^{t}\ddot{U}+\frac{\Delta t^3}{6}\,{}^{t}\dddot{U}+\ldots\ldots \tag{2}$$

$${}^{t+\Delta t}\dot{U}={}^{t}\dot{U}+\Delta t\,{}^{t}\ddot{U}+\frac{\Delta t^2}{2}\,{}^{t}\dddot{U}+\frac{\Delta t^3}{6}\,{}^{t}\ddddot{U}+\ldots\ldots \tag{3}$$

The above two equations are truncated and expressed in the following forms:

$${}^{t+\Delta t}U={}^{t}U+\Delta t\,{}^{t}\dot{U}+\frac{\Delta t^2}{2}\,{}^{t}\ddot{U}+\frac{\Delta t^3}{6}\,{}^{t}\dddot{U}+\alpha\Delta t^4\,{}^{t}\ddddot{U} \tag{4}$$

$$ {}^{t+\Delta t}\dot{U} = {}^{t}\dot{U} + \Delta t\, {}^{t}\ddot{U} + \frac{\Delta t^2}{2}\, {}^{t}\dddot{U} + \delta \Delta t^3\, {}^{t}\ddddot{U} \tag{5} $$

If the acceleration varies in quadratic way from t-Δt to t+Δt, then the following equations can be written:

$$ {}^{t}\dddot{U} = \frac{1}{2\Delta t}\left({}^{t+\Delta t}\ddot{U} - {}^{t-\Delta t}\ddot{U} \right) \tag{6} $$

$$ {}^{t}\ddddot{U} = \frac{1}{\Delta t^2}\left({}^{t-\Delta t}\ddot{U} - 2\,{}^{t}\ddot{U} + {}^{t+\Delta t}\ddot{U} \right) \tag{7} $$

Substituting equations (6) and (7) into equations (4) and (5), and rearranging the results produce:

$$ {}^{t+\Delta t}\dot{U} = {}^{t}\dot{U} + [(\delta - \frac{1}{4})\, {}^{t-\Delta t}\ddot{U} + (1-2\delta)\, {}^{t}\ddot{U} + (\delta + \frac{1}{4})\, {}^{t+\Delta t}\ddot{U}]\Delta t \tag{8} $$

$$ {}^{t+\Delta t}U = {}^{t}U + {}^{t}\dot{U}\,\Delta t + [(\alpha - \frac{1}{12})\, {}^{t-\Delta t}\ddot{U} + (\frac{1}{2} - 2\alpha)\, {}^{t}\ddot{U} + (\alpha + \frac{1}{12})\, {}^{t+\Delta t}\ddot{U}]\Delta t^2 \tag{9} $$

Here, equations (8) and (9) are used to approximate the velocity and displacement vectors at time t+Δt respectively. Although the Newmark method can be derived through similar procedure [17], more terms in the Taylor series expansion are kept in the proposed method. Later in Appendix A, it is shown that this strategy guarantees the second order accuracy for any values of δ and α. The parameters δ and α are introduced in order to improve accuracy and to obtain unconditional stability state for each time step. Special case is taken place when δ=1/4 and α=1/12 that leads to the presented method turning into the linear acceleration scheme. Equations (8) and (9) must be solved for ${}^{t+\Delta t}\ddot{U}$, meaning that calculation of ${}^{t+\Delta t}U$ and ${}^{t+\Delta t}\dot{U}$ require value of ${}^{t+\Delta t}\ddot{U}$. Therefore, the proposed method is considered as an implicit integration scheme. Equations (8) and (9) are in fact two-step integration schemes; meaning that the solution at time t+Δt depends on the solution at times t and t-Δt. Note that ${}^{0}U$ and ${}^{0}\dot{U}$ are known and ${}^{0}\ddot{U}$ can be calculated using equation (1) at time t=0. To start the time integration scheme, the solution at time Δt is required before applying equations (8) and (9). This can be computed by using any one-step methods such as the linear acceleration or the average acceleration methods. Once the value of the acceleration is obtained for Δt, the values for the next time steps are calculated using equations (8) and (9).

3. NUMERICAL STABILITY

To examine the stability of the new implicit method, the equation of motion for a single degree of freedom system at time $t+\Delta t$ is considered; as follows:

$$^{t+\Delta t}\ddot{x} + 2\xi\omega\,^{t+\Delta t}\dot{x} + \omega^2\,^{t+\Delta t}x = ^{t+\Delta t}r \tag{10}$$

where x is the displacement, $2\xi\omega = c/m$; $\omega^2 = k/m$; and $^{t+\Delta t}r = ^{t+\Delta t}p/m$. The natural frequency of vibration is ω and the time period of the motion is $T=2\pi/\omega$. If vector U is replaced by the single dependent variable x, then equations (8) and (9) are expressed as follows:

$$^{t+\Delta t}\dot{x} = ^{t}\dot{x} + [(\delta - \frac{1}{4})\,^{t-\Delta t}\ddot{x} + (1-2\delta)\,^{t}\ddot{x} + (\delta + \frac{1}{4})\,^{t+\Delta t}\ddot{x}]\Delta t \tag{11}$$

$$^{t+\Delta t}x = ^{t}x + ^{t}\dot{x}\Delta t + [(\alpha - \frac{1}{12})\,^{t-\Delta t}\ddot{x} + (\frac{1}{2} - 2\alpha)\,^{t}\ddot{x} + (\alpha + \frac{1}{12})\,^{t+\Delta t}\ddot{x}]\Delta t^2 \tag{12}$$

Substituting equations (11) and (12) into equation (10), one can find an equation with $^{t+\Delta t}\ddot{x}$ as the only unknown. Solving for $^{t+\Delta t}\ddot{x}$ and substituting into equations (11) and (12), $^{t+\Delta t}\dot{x}$ and $^{t+\Delta t}x$ are calculated. Thus, the following recursive relationship can be established.

$$\begin{Bmatrix} ^{t+\Delta t}\ddot{x} \\ ^{t}\ddot{x} \\ ^{t+\Delta t}\dot{x} \\ ^{t+\Delta t}x \end{Bmatrix} = [A] \begin{Bmatrix} ^{t}\ddot{x} \\ ^{t-\Delta t}\ddot{x} \\ ^{t}\dot{x} \\ ^{t}x \end{Bmatrix} + \{L\}\,^{t+\Delta t}r \tag{13}$$

The matrix $[A]$ and vector $\{L\}$ are called the "integration approximation" and "load" operators respectively. The coefficients of the above two matrix and vector for the proposed method are found in Appendix B. The spectral radius of $[A]$, $\rho(A)$, is defined by:

$$\rho(A) = \max_i |\lambda_i| \tag{14}$$

where λ_i are the eigenvalues of $[A]$. The presented method is stable if the eigenvalues of $[A]$ are not larger than one in modulus; which means $\rho(A) \le 1$. Moreover, it is also required that the eigenvalues of $[A]$ of multiplicity greater than one are strictly less than one in modulus [18]. Thus, in order to examine the stability of the proposed method for various values of δ and α, the spectral radii, ρ, are plotted versus $\omega.\Delta t$. Figures 1 to 3 show that for

$\delta \geq 1/3$ and $\alpha = \delta - 1/6$, the value of spectral radius becomes one. It is proven in Appendix C that the unconditional stability is reached in the following ranges:

$$\delta \geq \frac{1}{3} \quad ; \quad \frac{\delta}{2} \leq \alpha \leq \delta - \frac{1}{6} \tag{15}$$

It is obvious that for $\delta=1/3$, the unconditionally stable exists only if $\alpha=1/6$.

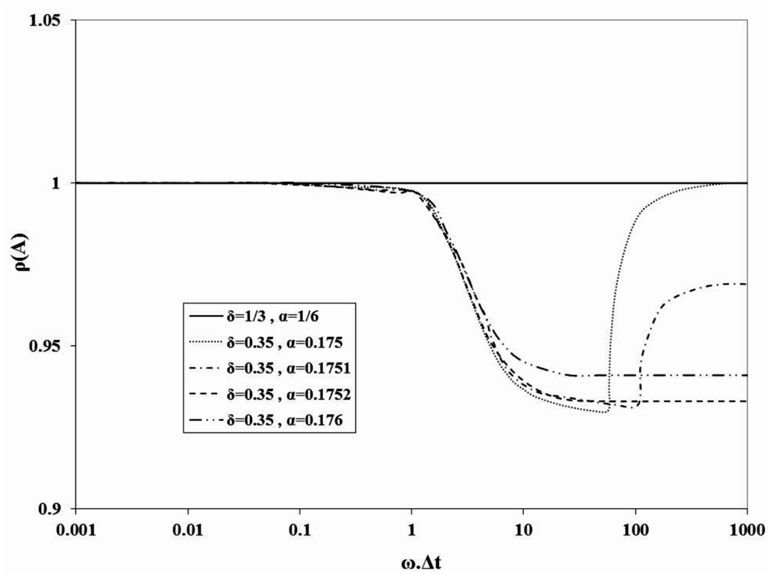

Figure 1. Spectral radii versus $\omega.\Delta t$ for various values of δ and α

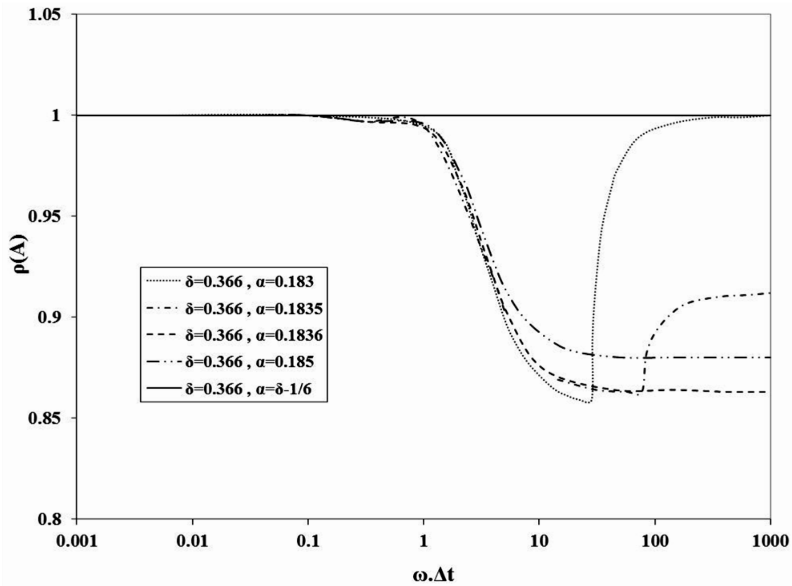

Figure 2. Spectral radii versus $\omega.\Delta t$ for various values of α and $\delta=0.366$

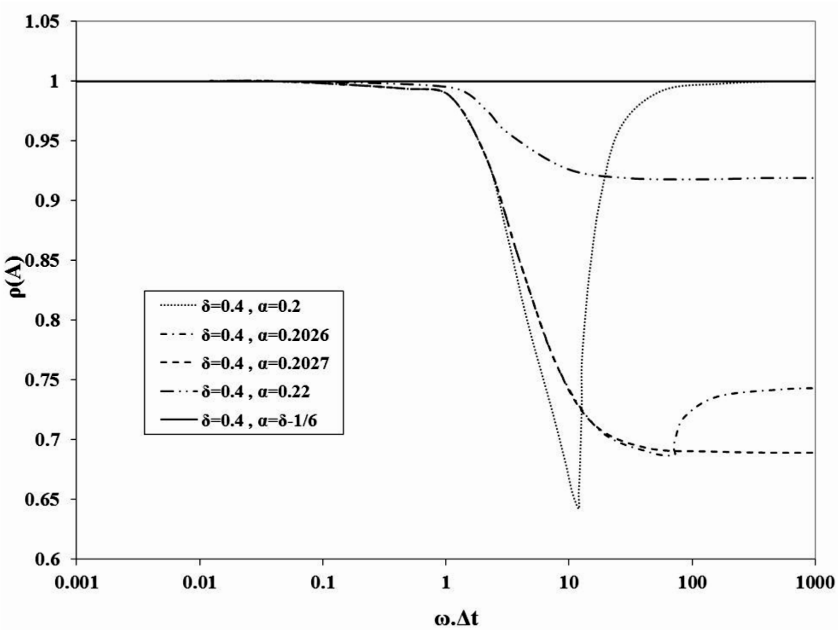

Figure 3. Spectral radii versus $\omega.\Delta t$ for various values of α and δ=0.4

4. ACUURACY OF THE PRSENTED METHOD

The accuracy of any numerical integration scheme is assessed by measuring the order of accuracy, numerical dissipation, and numerical dispersion of the method. In the new unconditionally stable method, any choices of parameters δ and α yield a second order accurate scheme (see Appendix A). The choice $\delta \geq 1/3$ with $\alpha = \delta - 1/6$ yields unconditionally stable methods with no numerical damping. However, to damp out any spurious participation of the higher modes, it is necessary to use algorithmic damping which can be introduced by selecting $\delta > 1/3$. For a fixed value of $\delta > 1/3$, the parameter α can control the amount of numerical dissipation in the higher modes. Maximal high frequency numerical dissipation is obtained when the value of $\rho_\infty = \lim_{\omega.\Delta t \to \infty}(A)$ is minimized [9]. For instance, for δ=0.35, the highest possible high frequency dissipation is obtained by selecting α=0.1752, as shown in Figure 1. Similarly, for values of δ=0.366 and δ=0.4, the minimum value of ρ_∞ is achieved by selecting α=0.1836 and α=0.2027 respectively as illustrated in Figures 2 and 3.

For various cases, numerical damping ratios are plotted versus $\Delta t/T$ in Figure 4. As illustrated in Figure 4, both choices of δ=0.366 with α=0.1836 and δ=0.4 with α=0.2027 maintain better accuracy in the low frequencies than the Wilson-θ method with θ=1.4 and the damped Newmark method[1] with β=0.3025 and γ=0.6. Figure 4 shows that in the proposed method, the higher modes are damped more effectively when δ=0.4 with α=0.2027 than by δ=0.366 with α=0.1836. It is important to note that numerical damping can be

[1] The equations describing the Newmark method are: ${}^{t+\Delta t}U = {}^{t}U + {}^{t}\dot{U}\,\Delta t + [(\frac{1}{2} - \beta)\,{}^{t}\ddot{U} + \beta\,{}^{t+\Delta t}\ddot{U}]\Delta t^2$
${}^{t+\Delta t}\dot{U} = {}^{t}\dot{U} + [(1-\gamma)\,{}^{t}\ddot{U} + \gamma\,{}^{t+\Delta t}\ddot{U}]\Delta t$

introduced in the presented scheme while maintaining its second order accuracy; whereas in the Newmark method, numerical damping can be introduced, although it reduces the level of accuracy from second order to first order [9].

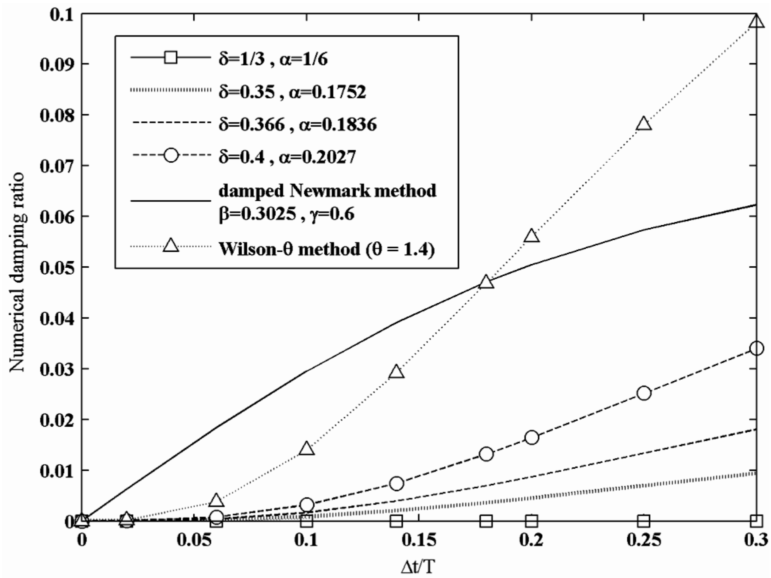

Figure 4. Numerical damping ratios for the Newmark, Wilson-θ and proposed methods

Relative period errors are also plotted versus $\Delta t/T$ in Figure 5 for various cases. The average acceleration method possesses the smallest period error of second order accurate, unconditionally stable linear multistep methods and thus its period errors may be used as a basis to compare the period errors of the numerically dissipative methods [19]. The least relative period error, while having unconditional stability, is obtained by selecting δ=1/3 and α=1/6. For this case, the proposed method and the average acceleration method have the same relative period errors as shown in Figure 5. This can be proven analytically through comparing the principal roots of the characteristic polynomial of the integration approximation operator in both methods. Despite the fact that selecting δ>1/3 increases period error, small period error with high dissipation is obtained by selecting a reasonable combination of parameters δ and α. For instance, engineering accuracy dictates that relative period error and amplitude decay[2] (per cycle) are less than 5 percent. In the new dissipative scheme, for δ=0.366 with α=0.1836 and δ=0.4 with α=0.2027, $\Delta t/T$ must be smaller than 0.115 and 0.107 respectively. However, for other dissipative methods such as α-method [20] with α=-0.3, Wilson-θ method with θ=1.4 [3], and Houbolt method [21], $\Delta t/T$ must be smaller than 0.1, 0.08, and 0.04 respectively [9]. Therefore, comparing the time step increment of all the well-known methods with the presented scheme indicates that the largest time step is permitted by the proposed method with values of δ=0.366 and α=0.1836. It means that in the proposed scheme a small number of time steps need to be used for engineering accuracy compared to the other dissipative methods mentioned above.

[2] For small numerical damping ratio, $\bar{\xi}$, the amplitude decay is $AD = 2\pi\bar{\xi}$

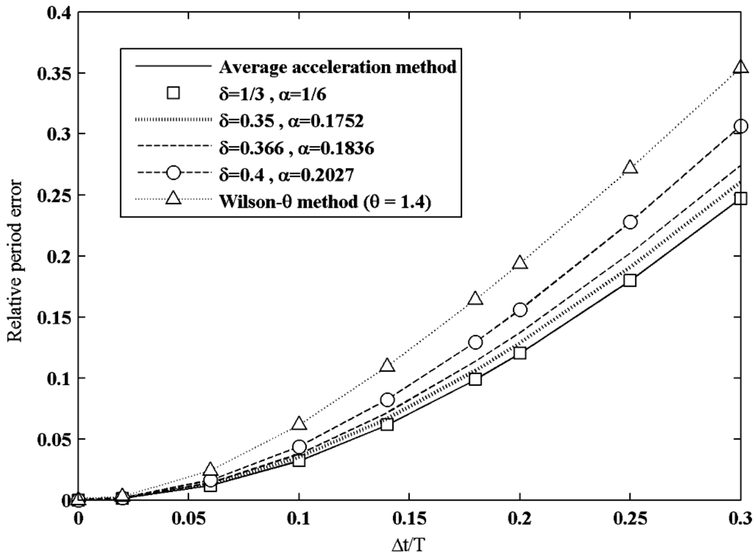

Figure 5. Relative period errors for the Average acceleration, Wilson-θ and proposed methods

5. SOLUTION PROCEDURE

Table 1 presents summary of the solution algorithm for the new implicit time integration scheme that can be used for computer programming. As indicated, once the stiffness, mass and damping matrices are calculated, one needs to select the time step increment along with the pertinent δ and α parameters. By applying the self starting scheme, the acceleration, velocity and displacement of the first two time steps are estimated. Then the constants of the integration is obtained; followed by assembly of the effective stiffness matrix. For the time step increment, the effective load vector is constructed and then from the effective stiffness and load matrices, the displacement vector is determined. Finally from the displacement vector, the corresponding acceleration and velocity vectors are obtained.

Table 1: Solution procedure for a linear elastic system

Input:

1. Input the stiffness matrix [k], mass matrix [m], and damping matrix [c]

2. Select time step Δt and parameters δ and α

$$\delta \geq \frac{1}{3} \quad ; \quad \frac{\delta}{2} \leq \alpha \leq \delta - \frac{1}{6}$$

3. Using a self starting procedure, calculate acceleration, velocity, and displacement vectors at the first two time steps

Calculations:

1. Calculate the integration constants

$$a_0 = \frac{1}{(\alpha + \frac{1}{12})\Delta t^2} \;,\; a_1 = \frac{(\delta + \frac{1}{4})}{(\alpha + \frac{1}{12})\Delta t} \;,\; a_2 = \frac{1}{(\alpha + \frac{1}{12})\Delta t}$$

$$a_3 = \frac{(\frac{1}{2} - 2\alpha)}{(\frac{1}{12} + \alpha)} \;,\; a_4 = \frac{(\alpha - \frac{1}{12})}{(\alpha + \frac{1}{12})} \;,\; a_5 = \frac{(\delta + \frac{1}{4})}{(\alpha + \frac{1}{12})} - 1$$

$$a_6 = (\frac{(\frac{1}{4} - \alpha)(4\delta + 1)}{(2\alpha + \frac{1}{6})} - 1 + 2\delta)\Delta t \;,\; a_7 = (\frac{(\alpha - \frac{1}{12})(\delta + \frac{1}{4})}{(\alpha + \frac{1}{12})} - \delta + \frac{1}{4})\Delta t$$

$$a_8 = (\delta - \frac{1}{4})\Delta t \;,\; a_9 = (1 - 2\delta)\Delta t \;,\; a_{10} = (\delta + \frac{1}{4})\Delta t$$

2. Form the effective stiffness matrix $[\hat{k}]$: $[\hat{k}] = [k] + a_0[m] + a_1[c]$

Time step:

1. Calculate the effective load vector at time $t + \Delta t$:

$${}^{t+\Delta t}\hat{R} = {}^{t+\Delta t}R + M(a_0\,{}^{t}U + a_2\,{}^{t}\dot{U} + a_3\,{}^{t}\ddot{U} + a_4\,{}^{t-\Delta t}\ddot{U}) + C(a_1\,{}^{t}U + a_5\,{}^{t}\dot{U} + a_6\,{}^{t}\ddot{U} + a_7\,{}^{t-\Delta t}\ddot{U})$$

2. Solve for the displacement vector at time $t + \Delta t$:

$$[\hat{k}]\,{}^{t+\Delta t}U = {}^{t+\Delta t}\hat{R}$$

3. Calculate the acceleration and velocity vectors at time $t + \Delta t$:

$${}^{t+\Delta t}\ddot{U} = a_0({}^{t+\Delta t}U - {}^{t}U) - a_2\,{}^{t}\dot{U} - a_3\,{}^{t}\ddot{U} - a_4\,{}^{t-\Delta t}\ddot{U}$$

$${}^{t+\Delta t}\dot{U} = {}^{t}\dot{U} + a_8\,{}^{t-\Delta t}\ddot{U} + a_9\,{}^{t}\ddot{U} + a_{10}\,{}^{t+\Delta t}\ddot{U}$$

6. BENCHMARK PROBLEMS

For the first benchmark problem, consider the following second order ordinary differential equation:

$$\ddot{x} + x = 0 \tag{16}$$

with initial conditions ${}^{0}x = 1$ and ${}^{0}\dot{x} = 0$. The exact solution of the above initial value problem is as follows:

$$x_{exact} = cos(t) \tag{17}$$

For the presented method, the following values of δ=1/3 and α=1/6 are used. Using

$\Delta t/T$=0.1 (T being the time period of the system which is 2π), the time step is Δt=0.2π. The error at a given time t is defined by the following relationship:

$$ {}^{t}e = \left| {}^{t}x - {}^{t}x_{exact} \right| \tag{18} $$

in which ${}^{t}x$ is the numerical solution at time t. Here this problem is solved using the average acceleration, Wilson-θ, and proposed methods and the corresponding errors are shown in Table 2. To start the new second-order accurate integration scheme, the average acceleration method is used in order to calculate the initial values at the first two time steps. Table 2 shows that the proposed scheme yields more accurate results compared to the other aforementioned methods. With the values selected for δ and α, the presented method and the average acceleration method do not yield the same results, even though both schemes have the same period error and they both have no numerical damping.

Table 2: Numerical solution of equation (16) using the average acceleration, Wilson-θ and presented methods for the first 10 time steps (Δt=0.2π)

Time	average acceleration method		Wilson-θ method (θ=1.4)		present method (δ=1/3 , α=1/6)	
	${}^{t}x$	${}^{t}e$	${}^{t}x$	${}^{t}e$	${}^{t}x$	${}^{t}e$
Δt	0.8203	0.0113	0.8187	0.0097	0.8203	0.0113
$2\Delta t$	0.3459	0.0369	0.3529	0.0439	0.3405	0.0315
$3\Delta t$	-0.2528	0.0562	-0.2273	0.0817	-0.2616	0.0474
$4\Delta t$	-0.7607	0.0483	-0.7220	0.0870	-0.7698	0.0392
$5\Delta t$	-0.9952	0.0048	-0.9651	0.0349	-1.0013	0.0013
$6\Delta t$	-0.8722	0.0632	-0.8785	0.0694	-0.8731	0.0641
$7\Delta t$	-0.4357	0.1267	-0.4968	0.1877	-0.4311	0.1221
$8\Delta t$	0.1573	0.1517	0.0464	0.2627	0.1658	0.1433
$9\Delta t$	0.6938	0.1152	0.5649	0.2441	0.7031	0.1059
$10\Delta t$	0.9810	0.019	0.8843	0.1157	0.9878	0.0122

For the second benchmark problem, consider the following two degrees of freedom system, which was solved using several direct integration methods in [9];

$$ \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{19} $$

With the initial conditions ${}^{0}U = \begin{Bmatrix} 1 \\ 10 \end{Bmatrix}$ and ${}^{0}\dot{U} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ where $k_1 = 10^4$, $k_2 = 1$, and $m_1 = m_2 = 1$.

The natural frequencies of this system are ω_1 =0.99995 and ω_2=100.005. Here assuming $\Delta t = T_1/20$ and T_1 being the period of the first mode, Δt=0.3 is selected. Using T_1 is due to the fact that in this example the first mode is physically more important than the second mode and here the second mode causes some undesirable oscillations.

The proposed scheme is compared with two other dissipative methods: the Wilson-θ method (with θ=1.4) and the damped Newmark method (with β=0.3025 and γ=0.6). The accuracy of the aforementioned methods is examined by the following two criteria: (i) Can the step by step integrators filter the higher mode oscillation from the response? (ii) Can the algorithms integrate the physically important oscillation accurately? For different time step number, results of the u_1 and u_2 displacements obtained from the Wilson-θ, Newmark and presented methods are all shown in Figures 6 to 11. Figure 6 shows that displacement u_1 obtained by the Wilson-θ method exhibits the overshoot phenomenon as indicated in [22], whereas displacement u_1 in the presented method overshoots only mildly (Figures 8 and 9). Although displacement u_1 obtained by the damped Newmark method (as shown in Figure 7) in the initial steps are close to the results obtained by the new unconditionally stable scheme, the proposed method managed to damp out the results much quicker than the damped Newmark method. Values of the displacement u_2 obtained by the Newmark, Wilson-θ, and presented schemes, with their corresponding errors, are illustrated in Table 3. As the results in the table indicate, the displacement u_2 calculated using the new dissipative scheme is more accurate than those obtained by the Newmark method or the Wilson-θ method.

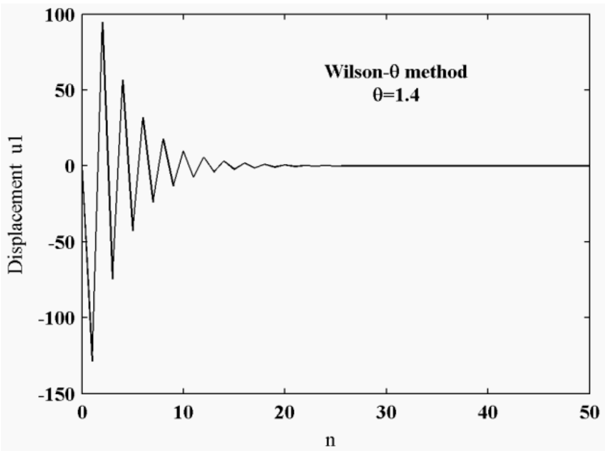

Figure 6. Displacement by the Wilson-θ method for the two degrees of freedom problem

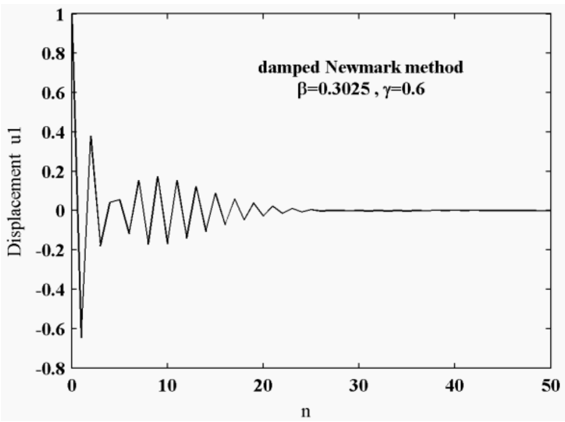

Figure 7. Displacement by the Newmark method for the two degrees of freedom problem

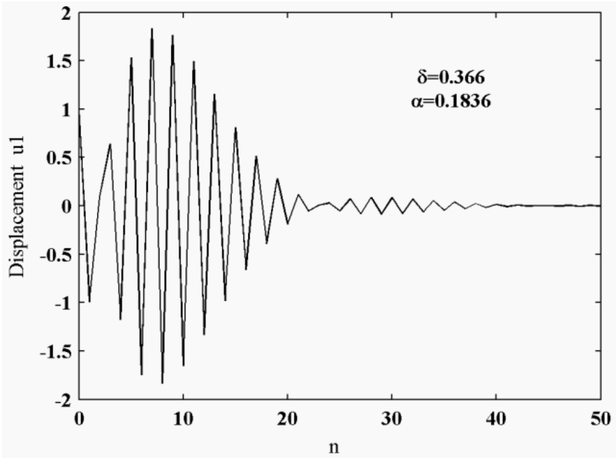

Figure 8. Displacement by the presented method for the two degrees of freedom problem (δ=0.366 and α=0.1836)

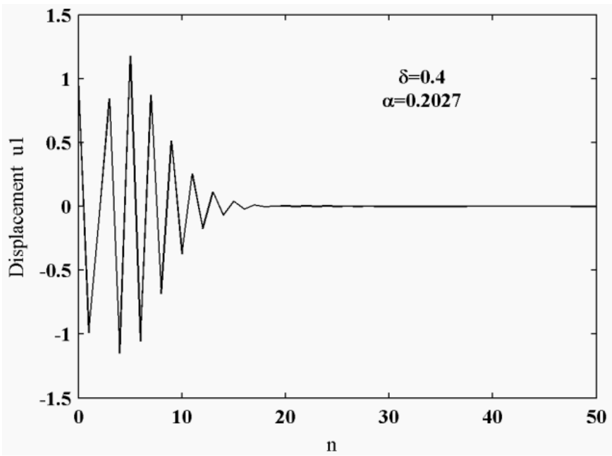

Figure 9. Displacement by the presented method for the two degrees of freedom problem (δ=0.4 and α=0.2027)

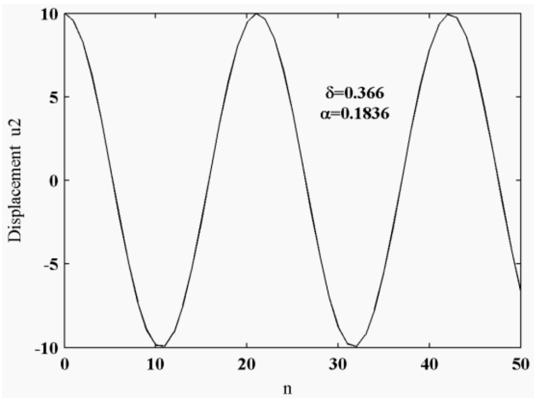

Figure 10. Displacement by the presented method for the two degrees of freedom problem (δ=0.366 and α=0.1836)

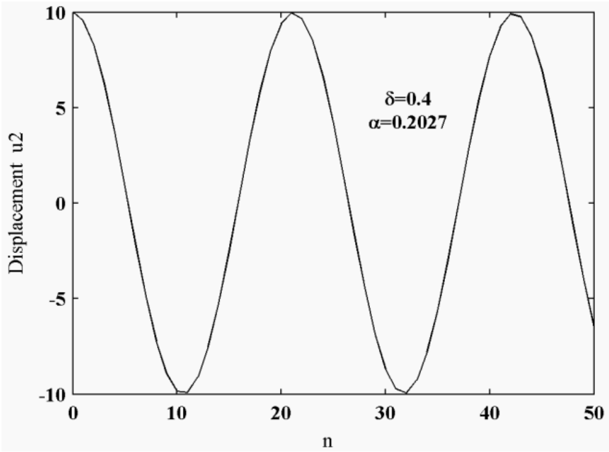

Figure 11. Displacement by the presented method for the two degrees of freedom problem (δ=0.4 and α=0.2027)

Table 3: Numerical solution of u_2 in equation (19) using the Newmark, Wilson-θ and presented methods for the first 20 time steps with Δt=0.3

Time	damped Newmark method (β=0.3025 , γ=0.6)		Wilson-θ method (θ=1.4)		proposed method (δ=0.366 , α=0.1836)	
	${}^{t}u_2$	${}^{t}e$	${}^{t}u_2$	${}^{t}e$	${}^{t}u_2$	${}^{t}e$
Δt	9.5621	0.0086	9.5722	0.0187	9.5601	0.0067
$2\Delta t$	8.2901	0.0364	8.2746	0.0209	8.2766	0.0229
$3\Delta t$	6.3032	0.0866	6.2986	0.0821	6.2670	0.0504
$4\Delta t$	3.7813	0.1572	3.7499	0.1258	3.7078	0.0837
$5\Delta t$	0.9504	0.2423	0.9021	0.1941	0.8231	0.1151
$6\Delta t$	-1.9391	0.3320	-2.0374	0.2337	-2.1329	0.1382
$7\Delta t$	-4.6334	0.4142	-4.7843	0.2632	-4.9020	0.1455
$8\Delta t$	-6.8981	0.4752	-7.1234	0.2498	-7.2399	0.1334
$9\Delta t$	-8.5387	0.5017	-8.8360	0.2044	-8.9428	0.0975
$10\Delta t$	-9.4168	0.4830	-9.7870	0.1128	-9.8601	0.0397
$11\Delta t$	-9.4624	0.4127	-9.8858	0.0108	-9.9125	0.0375
$12\Delta t$	-8.6785	0.2900	-9.1318	0.1634	-9.0945	0.1261
$13\Delta t$	-7.1412	0.1196	-7.5862	0.3253	-7.4790	0.2181
$14\Delta t$	-4.9918	0.0872	-5.3874	0.4828	-5.2068	0.3023
$15\Delta t$	-2.4239	0.3138	-2.7237	0.6136	-2.4784	0.3683
$16\Delta t$	0.3339	0.5388	0.1716	0.7011	0.4675	0.4052
$17\Delta t$	3.0382	0.7392	3.0493	0.7281	3.3718	0.4056
$18\Delta t$	5.4527	0.8921	5.6590	0.6858	5.9800	0.3648
$19\Delta t$	7.3683	0.9773	7.7762	0.5695	8.0629	0.2827
$20\Delta t$	8.6217	0.9794	9.2175	0.3836	9.4382	0.1629

7. CONCLUDING REMARKS

In this paper, a new implicit step by step integration scheme was presented for linear structural dynamics. The proposed method assumed quadratic variation of the acceleration at each time step whereas in the classical methods a linear function is normally implemented. The strategy of the proposed method is accomplished by employing the two parameters δ and α. Hence, a family of unconditionally stable schemes with high accuracy is introduced. For any value of δ and α, second order accuracy is guaranteed. The presented scheme can introduce numerical dissipation in the higher modes which is used intentionally in order to filter out the spurious high frequency components. The new dissipative method allows numerical damping while maintaining second order accuracy. It is important that numerical damping can be introduced in the proposed method without modifying the time discrete equation of motion; meaning that the fundamental equilibrium equations are exactly satisfied at the beginning and at the end of the time step. The proposed scheme also permits a parametric control of numerical dissipation in the higher modes. The adverse effects on the lower modes in the new second-order accurate scheme are less than other dissipative methods such as the Wilson-θ and the Newmark methods. Moreover, the presented method has smaller relative period error than other classical unconditionally stable schemes. In this paper, examples were provided in order to show the accuracy of the new implicit method as well as demonstrating the capability of effectively damping out the higher modes. Finally, although in this article the presented scheme is only used for the demonstration in the linear problems, it is expected that the technique can be extended in order to be implemented for the nonlinear problems.

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APPENDIX A–ORDER OF ACCURACY

To determine the order of accuracy, first, the exact solution is inserted into the equations (11) and (12). Then the discrete equation of motion is used to eliminate acceleration terms;

$$y(t+\Delta t) = y(t) + A_1 \dot{y}(t-\Delta t) + A_2 \dot{y}(t) + A_3 \dot{y}(t+\Delta t) + \Delta t.\tau(t) \quad \text{(A1)}$$

where

$$y(t)=\begin{Bmatrix} x(t) \\ \dot{x}(t) \end{Bmatrix} \quad , \quad \dot{y}(t)=\begin{Bmatrix} \dot{x}(t) \\ -2\xi\omega\,\dot{x}(t)-\omega^2 x(t)-r(t) \end{Bmatrix}$$

$$A_1=\begin{bmatrix} 0 & (\alpha-\frac{1}{12})\Delta t^2 \\ 0 & (\delta-\frac{1}{4})\Delta t \end{bmatrix} \quad , \quad A_2=\begin{bmatrix} \Delta t & (\frac{1}{2}-2\alpha)\Delta t^2 \\ 0 & (1-2\delta)\Delta t \end{bmatrix} \quad , \quad A_3=\begin{bmatrix} 0 & (\alpha+\frac{1}{12})\Delta t^2 \\ 0 & (\delta+\frac{1}{4})\Delta t \end{bmatrix}$$

and τ is called the vector of local truncation error. Note that, the symbols $x(t_n)$ and $\dot{x}(t_n)$ denote the exact values of displacement and velocity at time t_n respectively. Taylor series expansions of x and r about time t are employed to obtain the explicit expression for τ, as follows:

$$\tau=\begin{Bmatrix} \Delta t^3[\frac{1}{24}.\frac{d^4x}{dt^4}+\alpha(2\xi\omega.\frac{d^3x}{dt^3}+\omega^2.\frac{d^2x}{dt^2}-\frac{d^2r}{dt^2})]+.... \\ \Delta t^2[\frac{1}{6}.\frac{d^4x}{dt^4}+\delta(2\xi\omega.\frac{d^3x}{dt^3}+\omega^2.\frac{d^2x}{dt^2}-\frac{d^2r}{dt^2})]+.... \end{Bmatrix}=\begin{Bmatrix} \Delta t^3(\frac{1}{24}-\alpha).\frac{d^4x}{dt^4}+... \\ \Delta t^2(\frac{1}{6}-\delta).\frac{d^4x}{dt^4}+... \end{Bmatrix} \tag{A2}$$

Therefore here $\tau=o(\Delta t^2)$, meaning that the algorithm is second order accurate for all values of δ and α.

APPENDIX B – COEFFICIENTS

The “integration approximation” and “load” operators for the proposed method are:

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \quad , \quad L=\begin{Bmatrix} \frac{\beta}{\omega^2\Delta t^2} \\ 0 \\ \frac{\beta(\delta+\frac{1}{4})}{\omega^2\Delta t} \\ \frac{\beta(\alpha+\frac{1}{12})}{\omega^2} \end{Bmatrix} \tag{B1}$$

where

$$a_{11}=-(\frac{1}{2}-2\alpha)\beta-(2-4\delta)\kappa \ , \ a_{12}=-(\alpha-\frac{1}{12})\beta-(2\delta-\frac{1}{2})\kappa$$

$$a_{13}=\frac{1}{\Delta t}(-\beta-2\kappa) \ , \ a_{14}=\frac{1}{\Delta t^2}(-\beta)$$

$$a_{21}=1 \ , \ a_{22}=a_{23}=a_{24}=0$$

$$a_{31}=\Delta t\left[1-2\delta-(\frac{1}{2}-2\alpha)(\delta+\frac{1}{4})\beta-(1-2\delta)(2\delta+\frac{1}{2})\kappa\right]$$

$$a_{32}=\Delta t\left[\delta-\frac{1}{4}-(\delta+\frac{1}{4})(\alpha-\frac{1}{12})\beta-(\delta-\frac{1}{4})(2\delta+\frac{1}{2})\kappa\right]$$

$$a_{33}=1-\beta(\delta+\frac{1}{4})-2(\delta+\frac{1}{4})\kappa\ \ ,\ \ a_{34}=\frac{-\beta}{\Delta t}(\delta+\frac{1}{4})$$

$$a_{41}=\Delta t^2\left[\frac{1}{2}-2\alpha-(\frac{1}{2}-2\alpha)(\alpha+\frac{1}{12})\beta-(2-4\delta)(\alpha+\frac{1}{12})\kappa\right]$$

$$a_{42}=\Delta t^2\left[\alpha-\frac{1}{12}-(\alpha-\frac{1}{12})(\alpha+\frac{1}{12})\beta-(2\delta-\frac{1}{2})(\alpha+\frac{1}{12})\kappa\right]$$

$$a_{43}=\Delta t\left[1-(\alpha+\frac{1}{12})\beta-2(\alpha+\frac{1}{12})\kappa\right]\ \ ,\ \ a_{44}=1-(\alpha+\frac{1}{12})\beta$$

and

$$\beta=(\frac{1}{\omega^2\Delta t^2}+\frac{2\xi(\delta+\frac{1}{4})}{\omega\Delta t}+(\alpha+\frac{1}{12}))^{-1}\ \ ,\ \ \kappa=\frac{\xi\beta}{\omega\Delta t}$$

APPENDIX C – CONDITION OF STABILITY

To obtain stability criterion, by rewriting equation (12) we have:

$$^{t+\Delta t}x={}^{t}x+{}^{t}\dot{x}\,\Delta t+[(\alpha-\frac{1}{12})\ {}^{t-\Delta t}\ddot{x}+(\frac{1}{2}-2\alpha)\ {}^{t}\ddot{x}+(\alpha+\frac{1}{12})\ {}^{t+\Delta t}\ddot{x}]\,\Delta t^2 \tag{C1}$$

Similarly:

$$^{t+2\Delta t}x={}^{t+\Delta t}x+{}^{t+\Delta t}\dot{x}\,\Delta t+[(\alpha-\frac{1}{12})\ {}^{t}\ddot{x}+(\frac{1}{2}-2\alpha)\ {}^{t+\Delta t}\ddot{x}+(\alpha+\frac{1}{12})\ {}^{t+2\Delta t}\ddot{x}]\,\Delta t^2 \tag{C2}$$

Subtracting equation (C2) from equation (C1) and replacing ${}^{t+\Delta t}\dot{x}$ with the right hand side of equation (11) yields:

$$^{t+2\Delta t}x-2\,{}^{t+\Delta t}x+{}^{t}x=[(\delta-\alpha-\frac{1}{6})\ {}^{t-\Delta t}\ddot{x}+(3\alpha-2\delta+\frac{5}{12})\ {}^{t}\ddot{x}+(\delta-3\alpha+\frac{2}{3})\ {}^{t+\Delta t}\ddot{x}+(\alpha+\frac{1}{12})\,{}^{t+2\Delta t}\ddot{x}\,]\,\Delta t^2 \tag{C3}$$

The undamped and homogeneous time discrete equation of motion is used to eliminate acceleration terms in equation (C3) which yields:

$$(1+h(\alpha+\frac{1}{12}))\,{}^{t+2\Delta t}x+(-2+h(\delta-3\alpha+\frac{2}{3}))\,{}^{t+\Delta t}x+(1+h(3\alpha-2\delta+\frac{5}{12}))\,{}^{t}x+h(\delta-\alpha-\frac{1}{6})\,{}^{t-\Delta t}x=0 \tag{C4}$$

where $h=\omega^2(\Delta t)^2>0$. Equation (C4) is the difference-equation form of the proposed method. Assuming that this equation has a solution of the form $x=\lambda^t$, then ${}^{t+2\Delta t}x=\lambda^{t+2\Delta t}$, ${}^{t+\Delta t}x=\lambda^{t+\Delta t}$, ${}^{t}x=\lambda^{t}$ and ${}^{t-\Delta t}x=\lambda^{t-\Delta t}$. Making these substitutions and dividing by $\lambda^{t-\Delta t}$, the characteristic polynomial is obtained as follows:

$$\lambda^3+\frac{(-2+h(\delta-3\alpha+\frac{2}{3}))}{(1+h(\alpha+\frac{1}{12}))}\lambda^2+\frac{(1+h(3\alpha-2\delta+\frac{5}{12}))}{(1+h(\alpha+\frac{1}{12}))}\lambda+\frac{h(\delta-\alpha-\frac{1}{6})}{(1+h(\alpha+\frac{1}{12}))}=0 \tag{C5}$$

The characteristic polynomial roots (λ) are identical to the eigenvalues of the integration approximation operator $[A]$. Thus, it is required to have $|\lambda_i|\leq 1$ for stability. Consider cubic equation in general form:

$$\lambda^3+A\lambda^2+B\lambda+C=0 \tag{C6}$$

The Routh-Hurwitz criterion gives sufficient conditions for the roots of a characteristic polynomial, to be less than or equal to one in modulus. The Routh-Hurwitz criterion takes the form [9]:

(a). $1+A+B+C\geq 0$
(b). $3+A-B-3C\geq 0$
(c). $3-A-B+3C\geq 0$
(d). $1-A+B-C\geq 0$
(e). $1-B-C(C-A)\geq 0$

Applying these expressions to the characteristic polynomial (C5) yields:

(a). $h\geq 0$ (automatically satisfied)
(b). $h\geq 0$ (automatically satisfied)
(c). $\delta\geq\frac{1}{3}$
(d). $\alpha\geq\frac{\delta}{2}$
(e). $\alpha\leq\delta-\frac{1}{6}$

Consequently, unconditional stability is reached when $\delta\geq 1/3$ and $\delta/2\leq\alpha\leq\delta-1/6$.