ASIAN JOURNAL OF CIVIL ENGINEERING (BHRC) VOL. 16, NO. 7 (2015)
PAGES 977-999

ON THE APPLICATION OF WEAK FORM INTEGRAL EQUATIONS TO FREE VIBRATION ANALYSIS OF TALL STRUCTURES

H. Saffari[1] and M. Mohammadnejad[2*]
[1]Department of Civil Engineering, Faculty of Engineering, ShahidBahonar University of Kerman, Kerman, Iran
[2]Department of Civil Engineering, Birjand University of Technology, Birjand, Iran

Received:20 February 2015; **Accepted:**26 May 2015

ABSTRACT

In this paper, an analytical approach is presented for free vibration analysis of tall structures with various lateral resisting systems and variable properties along the height. Primarily, according to replacement beam theory, the tall structure is modeled by a non-prismatic Timoshenko beam with shear and global bending stiffness which is supported laterally by a beam with local bending stiffness. The vibration frequencies of shear beam, bending beam and as well as Timoshenko beam are calculated and the vibration frequencies of tall structure are obtained by combination of the obtained vibration frequencies. Presented analytical approach is also used to calculate the mode shape functions and internal forces of the tall structure. The efficiency and accuracy of the current approach are confirmed through comparison of the numerical results to those obtained using available finite-element software and other references.

Keywords: Tall structure; vibration frequency; Timoshenko beam; shear beam; bending beam; weak form integral equation.

1. INTRODUCTION

According to the structural dynamics theory, the tall structure can be approximately modeled by a non-prismatic cantilever beam. The approximate stiffness is considered for this cantilever beam. Therefore, the analysis is carried out on the replacement cantilever beam instead of original structure. After this modeling, the governing differential equations that describe the vibration behavior of tall structure should be solved. Kind of the governing differential equation, such as pure bending vibration, shear vibration or shear-bending vibration, depends on kind of the lateral load-resisting system of tall structure. Timoshenko

*E-mail address of the corresponding author:mohammadnejad@birjandut.ac.ir (M. Mohammadnejad)

beam equation governs on the vibration behavior of structures that have lateral load-resisting system with shear-bending stiffness (e.g. wall-frame structures). Timoshenko beam equation considers the bending and shear deformations along with the rotational inertia effects. For tall structures that have lateral load-resisting system with bending stiffness only (e.g. structures with shear-walls system), Bernoulli beam equation governs on their vibration behavior and for tall structures that have lateral load-resisting system with shear stiffness only, shear beam equation has best description for their vibration behavior.

Kazaz and Gülkan [1] developed a modified theory on the premise that a frame-wall system, deforming in shear and flexural modes, can be separated into two substructures that lie above and below the point of counter-flexure in the base story columns. Park et al. [2] proposed an analytical model for the dynamic analysis of tall buildings with a shear wall–frame structural system. They showed that the deformed shape of the shear wall–frame structural system is the combination of flexural mode and shear mode. Rahgozar et al. [3] presented a dynamic analysis of combined system of framed tube and shear walls by Galerkin method using B-spline functions.Kamgar and Saadatpour [4] determined the first natural frequency of tall buildings with a combined system of framed tube, shear core, belt truss and outrigger system with multiple jumped discontinuities in the cross section of framed tube and shear core under axial force.Malekinejad and Rahgozar [5] developed an analytical approach based on energy principles for computation of natural frequencies and mode shapes of multistory buildings constructed using framed tube, shear core and double belt trusses systems. Malekinejad and Rahgozar [6] presented the approximate formulas for dynamic response of tubular tall building structures. Zalka [7] presented the closed-form formulae for the torsional analysis of asymmetrical multi-storey buildings braced by moment-resisting frames, shear walls and cores. Mohammadnejad et al. [8] and Saffari et al. [9] presented an analytical approach for determination of the vibration frequencies of non-prismatic Bernoulli and Timoshenko beams by converting the governing differential equation to its weak-form integral equations. Lee [10] developed an approximate solution procedure for free vibration analysis of tube-in-tube tall buildings using the power-series solution method. Many researchers calculated the vibration frequencies of tall structures using various approaches [11-19].

In this paper an analytical approach is proposed for determination of the vibration frequencies of tall structures with variable properties along the height. According to “Replacement Beam” theory, the tall structure with various lateral load-resisting systems, such as bending frames, braced frames, shear walls, coupled shear walls and combination of them, is modeled by a non-prismatic Timoshenko beam with shear and global bending stiffnesses which is supported laterally by a beam with local bending stiffness.Using an analytical approach, the governing differential equations for free vibration of shear beam, bending beam and as well as Timoshenko beam are solved and corresponding vibration frequencies are calculated. The vibration frequencies of original structure are calculated by combination of the obtained vibration frequencies. Presented analytical approach is used to calculate the mode shape functions and internal forces of tall structure.

2. METHOD OF ANALYSIS

Based on the replacement beam theory, a tall structure with various lateral resisting systems including shear-wall, coupled shear wall, moment frames and braced frames can be modeled by a cantilever sandwich beam with variable properties along the height. The sandwich beam characterizes three kinds of stiffness: the global bending stiffness (D_0), the local bending stiffness (D_1) and shear stiffness (S). Every sandwich beam is equivalent to a Timoshenko beam (with global bending and shear stiffness) which is connected to a beam with local bending stiffness by axially rigid connections (Fig. 1). These three types of stiffness correspond to three types of deformation included to structure. The deformation of structure is obtained using a combination of these three deformations [11,12,13].

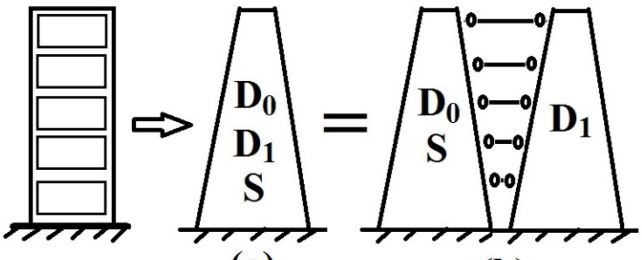

Figure 1. (a) Replacement beam of the frame; (b) the sandwich beam is equivalent to the Timoshenko beamSupported by a beam with bending deformation only

The tall structureis modeled by the replacement beam; therefore, the global and local bending stiffness as well as shear stiffness of lateral resisting systems of original structure are calculated at each story. Using approximate relations, these stiffnesses are combined in order to reach the global and local bending stiffness as well as shear stiffness of the replacement beam. Interpolated functions which optionally define the variation of stiffness in height of the structure are applied in the course of calculation. Furthermore, these functions are used for calculation of structural frequencies when the governing equations are transformed into solvable equations. Therefore, we obtain the vibration frequencies of shear beam, bending beam and Timoshenko beam and combine them to calculate the vibration frequencies of original structure.

3. VIBRATION FREQUENCY OF SHEAR BEAM ω_m^S

It has been recognized that the lateral deflection of most buildings is not purely flexural, but there is a considerable contribution from shear deflections in most cases. If shear deformation is dominated in the total deformation of buildings in their horizontal vibrations, such structures are usually called shear-type buildings [20]. Therefore, shear beam equation has best description for vibration behavior such structures.The governing differential equation for free vibration of a non-prismatic shear beam and shear force acting on section of the beam is givenas follows [20]:

$$\begin{cases} \frac{\partial}{\partial x}\left(K_S(x)\frac{\partial}{\partial x}\vartheta(x,t)\right)-m(x)\frac{\partial^2}{\partial t^2}\vartheta(x,t)=0 \\ V(x,t)=K_S(x)\frac{\partial}{\partial x}\vartheta(x,t) \end{cases} \tag{1}$$

In which $K_S(x)$, $m(x)$, $\vartheta(x,t)$ and $V(x,t)$ are shear stiffness function, the mass per unit length, transverse displacement and shear force acting on section of the beam, respectively. A harmonic vibration is assumed which defines the lateral displacement of the beam as $\vartheta(x,t)=\phi(x)e^{i\left(\omega_m^S\right)t}$. $\phi(x)$ and ω_m^S are themode shape function and vibration frequency of shear beam with mass per unit length. Substitution of this relationship into relations (1) leads to a single-variable equation in terms of location, as follows:

$$\frac{d}{d\xi}\left(K_S(\xi)\frac{d}{d\xi}\phi(\xi)\right)+\lambda m(\xi)\phi(\xi)=0 \tag{2}$$

$$V(\xi)=\frac{K_S(\xi)}{L}\frac{d}{d\xi}\phi(\xi) \tag{3}$$

In the above relation, it is assumed that $\xi=\frac{x}{L}$ and $\lambda=\left(\omega_m^S\right)^2 L^2$ in which L is the beam length. Eq. (2) is, in fact, the free vibration equation of a non-prismatic shear beam based on the non-dimensional variable ξ. In order to convert Eq. (2) to its weak form, we integrate from both sides of Eq. (2) twice with respect to ξ within the range 0 to ξ. The result is the integral equations as follows:

$$K_S(\xi)\frac{d}{d\xi}\phi(\xi)+\lambda\int_0^{\xi} m(s)\phi(s)ds=C_1 \tag{4}$$

$$\int_0^{\xi}\left[\lambda(\xi-s)m(s)-K_S'(s)\right]\phi(s)ds+K_S(\xi)\phi(\xi)=C_1\xi+C_2 \tag{5}$$

In Eq. (5) C_1 and C_2 are the integration constants which are determined through boundary conditions of both ends of the beam. Eq. (5) is the integral equation of the weak form for free vibration of a non-prismatic shear beam. Eqs. (4-5) are applicable for determination of the integration constants. Further, substitution of the resulting integration constants into Eq. (5) yields an integral equation in $\phi(\xi)$. The tall building is modeled by a cantilever beam. Therefore, considering Eq. (3), the following relations are defined for boundary conditions in a shear cantilever beam:

$$\begin{cases} \xi = 0 & \phi = 0 \\ \xi = 1 & V = 0 \rightarrow \left[\dfrac{K_S(\xi)}{L} \dfrac{d}{d\xi} \phi(\xi) \right]_{\xi=1} = 0 \end{cases} \tag{6}$$

By applying the above boundary conditions to Eqs. (4-5), the integration constants are determined.

Substitution of the integration constants obtained into Eq. (5) yields an integral equation as follows:

$$K_S(\xi)\phi(\xi) + \int_0^{\xi} f_1(\xi,s)\phi(s)ds + \int_0^{1} f_2(\xi,s)\phi(s)ds = 0 \tag{7}$$

In Eq. (7) the functions $f_1(\xi, s)$ and $f_2(\xi, s)$ are expressed by the following relations:

$$\begin{cases} f_1(\xi,s) = \lambda(\xi - s)m(s) - K_S'(s) \\ f_2(\xi,s) = -\lambda \xi m(s) \end{cases} \tag{8}$$

4. VIBRATION FREQUENCY OF SHEAR BEAM WITH A LUMPED MASS AT FREE END ω_{Mm}^{S}

In this section, it is assumed that there is a lumped mass at free end of the cantilever shear beam. The rotational inertia effects and weight of lumped mass have been neglected. The analysis method is exactly the same as what was stated in section 3. The difference, however, is that the shear force is of non-zero value at the free end of the beam (Fig. 2). Other boundary conditions are assumed unchanged. In this case, the boundary conditions are stated as follows [21]:

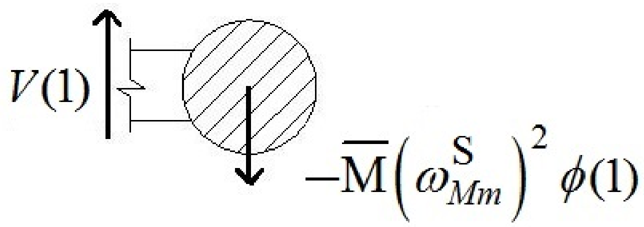

Figure 2. shear force acting on the end mass of the cantilever shear beam

$$\begin{cases} \xi = 0, & \phi = 0 \\ \xi = 1 & V = -\overline{\mathrm{M}}\left(\omega_{Mm}^{S}\right)^2 \phi(1) \quad or \quad \left[\dfrac{K_S(\xi)}{L} \dfrac{d}{d\xi} \phi(\xi) \right]_{\xi=1} = -\overline{\mathrm{M}}\left(\omega_{Mm}^{S}\right)^2 \phi(1) \end{cases} \tag{9}$$

In which $\overline{\mathrm{M}}$ is the lumped mass at free end of the beam and ω_{Mm}^{S} is vibration frequency of the shear beam with mass per unit length $m(\xi)$ and lumped mass at free end. By applying the above boundary conditions to Eqs.(4-5) the integration constants are determined. Substitution of the integration constants obtained into Eq. (5) yields an integral equation as follows:

$$K_S(\xi)\phi(\xi)+\int_0^{\xi} f_1(\xi,s)\phi(s)ds+\int_0^{1} f_2(\xi,s)\phi(s)ds=0 \qquad (10)$$

In Eq. (10) the functions $f_1(\xi,s)$ and $f_2(\xi,s)$ are expressed by the following relations:

$$\begin{cases} f_1(\xi,s)=\lambda(\xi-s)m(s)-K_S'(s) \\ f_2(\xi,s)=\lambda\xi m(s)\left[\beta s-1\right]+\beta\xi K_S'(s) \end{cases} \qquad (11)$$

In which $\beta=\dfrac{L\bar{M}\left(\omega_{Mm}^{S}\right)^2}{L\bar{M}\left(\omega_{Mm}^{S}\right)^2-K_S(1)}$ is applied.

The functions (11) are used to calculatethevibration frequencies ω_{Mm}^{S} but, the first vibration frequency of shear beam with mass per unit length and lumped mass at free end ω_{1Mm}^{S} can be also calculated by combination of the vibration frequencies ω_{1m}^{S} and ω_{1M}^{S} according to Dunkerley's theorem. Dunkerley's theorem is presented in section 11. ω_{1m}^{S} is the first vibration frequency of shear beam with mass per unit length (section 3) and ω_{1M}^{S} is the first vibration frequency of shear beam with lumped mass when mass per unit length of the beam is neglected.By setting $m(s)=0$ in relations (11)we obtain the relations which can be used to calculate ω_{M}^{S} . It should be noted that a shear beam with lumped mass at free end and without mass per unit length vibrates as a single-degree of freedom system; therefore it only has one vibration frequency.

5. VIBRATION FREQUENCY OF BENDING BEAM ω_m^B

Neglecting damping terms, the governing differential equation for free vibration of a non-prismatic bending beam (Bernoulli beam) is given by [21]:

$$\frac{\partial^2}{\partial x^2}\left[K_B(x)\frac{\partial^2}{\partial x^2}\vartheta(x,t)\right]+m(x)\frac{\partial^2}{\partial t^2}\vartheta(x,t)=0 \qquad (12)$$

In which $K_B(x)=EI(x)$ is bending stiffness function which depends on both young's modulus E and the inertial moment of cross-sectional area $I(x)$. $m(x)$ and $\vartheta(x,t)$ are the mass per unit length and transverse displacement, respectively. Saffari et al. [9] have solved the governing differential equation for free vibration of cantilever bending beam under variable axial forces. Here we neglect the effects of axial forces. Therefore, $N(\xi)=0$ ($N(\xi)$ is axial force function) is substituted into the obtained relations. This results in an integral equation as follows:

$$K_B(\xi)\phi(\xi)+\int_0^{\xi} f_1(\xi,s)\phi(s)\mathrm{d}s+\int_0^{1} f_2(\xi,s)\phi(s)\mathrm{d}s=0 \tag{13}$$

In equation (11), functions $f_1(\xi,s)$ and $f_2(\xi,s)$ are expressed by the following relations:

$$\begin{cases} f_1(\xi,s)=(\xi-s)K_B''(s)-2K_B'(s)-\dfrac{\lambda}{6}(\xi-s)^3 m(s) \\ f_2(\xi,s)=\dfrac{\lambda}{6}\xi^3 m(s)-\dfrac{\lambda}{2}\xi^2 s m(s) \end{cases} \tag{14}$$

In which $\lambda=(\omega_m^B)^2 L^4$ and ω_m^B is the vibration frequency of bending beam with mass per unitlength $m(\xi)$.

6. VIBRATION FREQUENCY OF BENDING BEAM WITH A LUMPED MASS AT FREE END ω_{Mm}^B

In this section, it is assumed that there is a lumped mass at free end of the cantilever bending beam. The rotational inertia effects and weight of lumped mass have been neglected. Saffari et al. [9] have solved the governing differential equation of bending beam with lumped mass at free end. By setting $N(\xi)=0$ in the obtained relations, the following integral equation is obtained:

$$K_B(\xi)\phi(\xi)+\int_0^{\xi} f_1(\xi,s)\phi(s)\mathrm{d}s+\int_0^{1} f_2(\xi,s)\phi(s)\mathrm{d}s=0 \tag{15}$$

In which the functions $f_1(\xi,s)$ and $f_2(\xi,s)$ can be stated as follows:

$$\begin{cases} f_1(\xi,s)=-\dfrac{\lambda}{6}(\xi-s)^3 m(s)+(\xi-s)K_B''(s)-2K_B'(s) \\ f_2(\xi,s)=-\dfrac{\alpha_2}{2}\xi^2 m(s)-\xi^2 g_1(s)-\left[\dfrac{\alpha_1}{6}\xi^3+\dfrac{\alpha_3}{2}\xi^2\right]g_2(s) \end{cases} \tag{16}$$

Where:

$$\begin{cases} g_1(s) = (1-s)K_B''(s) - 2K_B'(s) - \dfrac{\lambda}{6}(1-s)^3 m(s) \\ g_2(s) = 6g_1(s) - \left(\dfrac{6LK_B(1)}{\overline{\mathrm{M}}} + 3\lambda(s-1)\right)m(s) \end{cases} \tag{17}$$

And

$$\begin{cases} \alpha_1 = \dfrac{\overline{\mathrm{M}}\left(\omega_{Mm}^B\right)^2 L^3}{6K_B(1) - 2\overline{\mathrm{M}}\left(\omega_{Mm}^B\right)^2 L^3} \\ \alpha_2 = -\dfrac{2K_B(1)L}{\overline{\mathrm{M}}} \\ \alpha_3 = \dfrac{6K_B(1) + \overline{\mathrm{M}}\left(\omega_{Mm}^B\right)^2 L^3}{6\overline{\mathrm{M}}\left(\omega_{Mm}^B\right)^2 L^3 - 18K_B(1)} \end{cases} \tag{18}$$

In which $\lambda = (\omega_{Mm}^B)^2 L^4$ and ω_{Mm}^B is the vibration frequency of bending beam with lumped mass and $\overline{\mathrm{M}}$ is the lumped mass at free end of the beam. The relations (16) can be used to calculate the vibration frequencies ω_{Mm}^B but, the first vibration frequency of the bending beam with mass per unit length and lumped mass at free end ω_{1Mm}^B can be calculated by another method. We can combine the vibration frequencies ω_{1m}^B and ω_{1M}^B according to Dunkerley's theoremand calculate ω_{1Mm}^B. Dunkerley's theorem is presented in section 11. ω_{1m}^B is the first vibration frequency ofbending beam with mass per unit length (section 5) and ω_{1M}^B is the first vibration frequency of the bending beam with lumped mass when mass per unit length of the beam is neglected. By setting $m(\xi) = 0$ in relations (16-18) we obtain the relations which can be used to calculate ω_M^B. It should be noted that a bending beam with lumped mass at free end and without mass per unit length vibrates as a single-degree of freedom system; therefore it only has one vibration frequency.

7. TRANSFORMATION OF THE INTEGRAL EQUATIONS OBTAINED INTO THE SYSTEM OF LINEAR ALGEBRAIC EQUATIONS

In the preceding sections, for shear and bending beams, we have converted the governing partial differential equations to the corresponding integral equations of the weak form. For integral equations (7-10-13-15), many existing techniques may be employed to determine the numerical solution or the approximate solution. In this paper, it is sufficient to determine characteristic values of the resulting integral equation, which is related to natural frequencies of free vibration of beam. Here we expand $\phi(\xi)$ as following power series:

$$\phi(\xi) = \sum_{r=0}^{R} c_r \xi^r \tag{19}$$

Where c_r are unknown coefficients and R is a given positive integer, which is adopted such that the accuracy of the results is sustained. Introducing Eq. (19) into integral equations obtained before leads to:

$$\sum_{r=0}^{R} \left[K_S(\xi)\xi^r + \int_0^{\xi} f_1(\xi,s)s^r \mathrm{ds} + \int_0^{1} f_2(\xi,s)s^r \mathrm{ds} \right] c_r = 0 \tag{20}$$

The function $K_B(\xi)$ replaces $K_S(\xi)$ when shear vibration is investigated. Both sides of (20) are multiplied by ξ^m and integrated subsequently with respect to ξ between 0 and 1. This results in a system of linear algebraic equations in c_r:

$$\sum_{r=0}^{R} \left[G(m,r) + F_1(m,r) + F_2(m,r) \right] c_r = 0 \qquad m = 0,1,2,...,R \tag{21}$$

In which functions $G(m,r)$, $F_1(m,r)$ and $F_2(m,r)$ are expressed as follows:

$$\begin{cases} G(m,r) = \int_0^1 \xi^{r+m} K_S(\xi) \mathrm{d}\xi \\ F_1(m,r) = \int_0^1 \int_0^{\xi} f_1(\xi,s) s^r \xi^m \,\mathrm{dsd}\xi \\ F_2(m,r) = \int_0^1 \int_0^1 f_2(\xi,s) s^r \xi^m \,\mathrm{dsd}\xi \end{cases} \tag{22}$$

The system of linear algebraic Eq. (21) may be expressed in matrix notations as follows:

$$[A]_{(R+1)\times(R+1)} [C]_{(R+1)\times 1} = [0]_{(R+1)\times 1} \tag{23}$$

In which $[A]$ and $[c]^T$ are coefficients matrix and unknowns vector transpose, respectively. In order to obtain the circular natural frequencies of the beam, functions $f_1(\xi,s)$ and $f_2(\xi,s)$ are first obtained. Introducing these functions into (22), the functions $G(m,r)$, $F_1(m,r)$ and $F_2(m,r)$ associated with the coefficients of matrix [A] are obtained next. The unknown parameter in the coefficients matrix [A] is therefore the circular natural frequency of the beam. $[c]=0$ is a trivial solution for the resulting system of equations introduced in (21).The natural frequencies are determined through calculation of a non-trivial solution for resulting system of equations. To achieve this, the determinant of the coefficients matrix of the system has to be vanished. Accordingly, a frequency equation in ω (which is a polynomial function of the order $2(R+1)$) is introduced. Given the fact that the mode shape function is approximated by the power series (19), the results' accuracy are improved if more number of the series sentences are taken into account.

8. VIBRATION FREQUENCY OF TIMOSHENKO BEAM ω_m^T

Timoshenko beam equation governs on the vibration behavior of structures that have lateral load-resisting system with shear-bending stiffness (e.g. wall-frame structures). Timoshenko beam equation considers the bending and shear deformations along with the rotational inertia effects. Taking the effect of shear deformation and rotational inertia into account, the governing differential equations for free vibration of non-prismatic Timoshenko beams are introduced as follows [21]:

$$\begin{cases} \frac{\partial}{\partial x}\left(kA(x)G\left[\theta(x,t)-\frac{\partial}{\partial x}\vartheta(x,t)\right]\right)+m(x)\frac{\partial^2}{\partial t^2}\vartheta(x,t)=0 & (24-a) \\ \frac{\partial}{\partial x}\left[D(x)\frac{\partial}{\partial x}\theta(x,t)\right]-kA(x)G\left[\theta(x,t)-\frac{\partial}{\partial x}\vartheta(x,t)\right]-m(x)r^2(x)\frac{\partial^2}{\partial t^2}\theta(x,t)=0 & (24-b) \end{cases}$$

In which $A(x), r(x), D(x), m(x), \vartheta(x,t), \alpha(x,t), G$ and k are the cross sectional area, the radius of gyration $r^2(x)=\frac{I(x)}{A(x)}$, the bending stiffness $D(x)=EI(x)$, the mass per unit length, the transverse displacement ofthe beam, the rigid rotation ofthe beam section, the modulus of elasticity in shear and the shear correction factor.Mohammadnejad et al [8] have solved equations (24). We here use their obtained results to calculate ω_m^T. The functions $f_1(\xi,s,r)$ and $f_2(\xi,s,r)$ have been calculated as follows:

$$\begin{cases} f_1(\xi,s,r)=\left[\frac{\lambda L}{2}(\xi-s)^2+\frac{L}{2}R(s)G(s)(\xi-s)^2-LD'(s)G(s)(\xi-s)+LD(s)G(s)\right]\int_s^1 m(z)z^r dz \\ \qquad +\left[-\frac{R'(s)}{2}(\xi-s)^2+D''(s)(\xi-s)+R(s)(\xi-s)-2D'(s)\right]s^r \\ f_2(\xi,s,r)=-\frac{A}{2}\xi^2 h_1(s,r)+B\,m(s)\xi^2 s^r+C\,m(s)\xi s^r \end{cases} \qquad (25)$$

The vibration frequencies of Timoshenko beam with mass per unit length and lumped mass at free end ω_{Mm}^T are calculated by combination of the vibration frequencies ω_{Mm}^B and ω_{Mm}^S according to Foppl theory. Foppl theory is presented in section 11.

9. MODE SHAPE FUNCTIONS

After calculation of the vibration frequencies according to presented approach, we can calculate the mode shape functions of the vibration. Given the fact that the mode shape function is approximated by the following power series:

$$\phi(\xi) = \sum_{r=0}^{R} c_r \xi^r = c_0 + c_1\xi + c_2\xi^2 + ... + c_R \xi^R \qquad (26)$$

The mode shape function of ith mode $\phi_i(\xi)$ corresponding to vibration frequency of ith mode ω_i is obtained as follows:

$$\phi_i(\xi) = \sum_{r=0}^{R} (c_r)_i \xi^r = (c_0)_i + (c_1)_i \xi + (c_2)_i \xi^2 + ... + (c_R)_i \xi^R \qquad (27)$$

In which $(C_r)_i$ (r=0,1,…R) are the unknown coefficients of the power series corresponding to ith mode. To calculate the mode shape function $\phi_i(\xi)$, the unknown coefficients of power series $(C_r)_i$ should be calculated independently. System of linear algebraic equations (23) has the matrix form $[A]_{(R+1,R+1)}[C_r]_{(R+1,1)} = 0$. The vibration frequency ω is the unknown parameter in the coefficients matrix $[A]_{(R+1,R+1)}$. By substitution of the vibration frequency of ith mode ω_i into the coefficients matrix $[A]_{(R+1,R+1)}$, the system of linear algebraic equations (23) takes the following form:

$$[\mathrm{A}_i]_{(R+1,R+1)}[C_{r_i}]_{(R+1,1)} = 0 \qquad or \qquad \begin{bmatrix} (\mathrm{A}_{1,1})_i & (\mathrm{A}_{1,2})_i & .... & (\mathrm{A}_{1,R+1})_i \\ (\mathrm{A}_{2,1})_i & (\mathrm{A}_{2,2})_i & .... & (\mathrm{A}_{2,R+1})_i \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ (\mathrm{A}_{R+1,1})_i & (\mathrm{A}_{R+1,2})_i & .... & (\mathrm{A}_{R+1,R+1})_i \end{bmatrix} \begin{bmatrix} (C_0)_i \\ (C_1)_i \\ \vdots \\ \vdots \\ (C_R)_i \end{bmatrix} = 0 \qquad (28)$$

In which $[\mathrm{A}_i]_{(R+1,R+1)}$ and $[C_{r_i}]_{(R+1,1)}$ are the coefficients matrix and unknowns vector corresponding to ith mode, respectively. In order to solve the system of linear algebraic equations (28) and calculate the unknowns vector $[C_{r_i}]_{(R+1,1)}$, we adopt $(C_0)_i = 1$ and substitute it into the system of linear algebraic equations (28). The result is as follows:

$$\overline{[A_i]}_{(R+1,R)} \overline{[C_{r_i}]}_{(R,1)} = \overline{[B_i]}_{(R+1,1)} \qquad (29)$$

In which:

$$\overline{[C_{r_i}]}_{(R,1)} = \begin{bmatrix} (C_1)_i \\ (C_2)_i \\ \vdots \\ (C_R)_i \end{bmatrix} \quad \text{and} \quad \overline{[B_i]}_{(R+1,1)} = \left[-\text{first column of coefficients matrix } [\mathrm{A}_i]\right] = \begin{bmatrix} -(\mathrm{A}_{1,1})_i \\ -(\mathrm{A}_{2,1})_i \\ \vdots \\ \vdots \\ -(\mathrm{A}_{R+1,1})_i \end{bmatrix} \tag{30}$$

The coefficientsmatrix $\overline{[A_i]}_{(R+1,R)}$ can be calculated by elimination of the first column of coefficients matrix $[A_i]_{(R+1,R+1)}$ as:

$$\overline{[A_i]}_{(R+1,R)} = \begin{bmatrix} (\mathrm{A}_{1,2})_i & (\mathrm{A}_{1,3})_i & \cdots\cdots\cdots & (\mathrm{A}_{1,\mathrm{R}+1})_i \\ (\mathrm{A}_{2,2})_i & (\mathrm{A}_{2,3})_i & \cdots\cdots\cdots & (\mathrm{A}_{2,R+1})_i \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ (\mathrm{A}_{R+1,2})_i & (\mathrm{A}_{R+1,3})_i & \cdots\cdots\cdots & (\mathrm{A}_{R+1,\mathrm{R}+1})_i \end{bmatrix} \tag{31}$$

Solving the system of linear algebraic equations (29) results the unknown coefficients of power series $(C_r)_i$ (r=1,…R). By substitution of the coefficients $(C_r)_i$ into the power series (27), the mode shape function corresponding to ith mode is calculated.

10. INTERNAL FORCES

In the response modal analysis of buildings subjected to earthquakes an equivalent load is determined in each mode of vibration (Fig. 3). For in-plane vibration, when the ground motion is in the plane of the vibration, the horizontal force of ith mode is [22]:

$$f_i(x) = \frac{\int_0^H m(x)\phi_i(x)dx}{\int_0^H m(x)\phi_i^2(x)dx} m(x)\phi_i(x)S_{Ai} \quad or \quad f_i(\xi) = \frac{\int_0^1 m(\xi)\phi_i(\xi)d\xi}{\int_0^1 m(\xi)\phi_i^2(\xi)d\xi} m(\xi)\phi_i(\xi)S_{Ai} \tag{32}$$

Where S_{Ai} is the spectral acceleration of ith mode (which depends on the period of vibration, damping and the ground peak acceleration), and H is the height of structure. The shear force acting on the cross section of the structure in ith mode is obtained by integrating Eq. (32) between x and H (or ξ and 1) (Fig. 3). Integration of Eq. (42) gives:

$$V_i(x) = \int_x^H f_i(x)dx \quad or \quad V_i(\xi) = H\int_\xi^1 f_i(\xi)d\xi \tag{33}$$

The total horizontal load, which is identical to the base shear force V_i, is obtained by setting $x = 0$ in Eq. (33).The base overturning moment can be calculated from the horizontal load Eq. (32) as follows (Fig. 3):

$$M_i = \int_0^H xf_i(x)dx \quad or \quad M_i = H^2\int_0^1 \xi f_i(\xi)d\xi \tag{34}$$

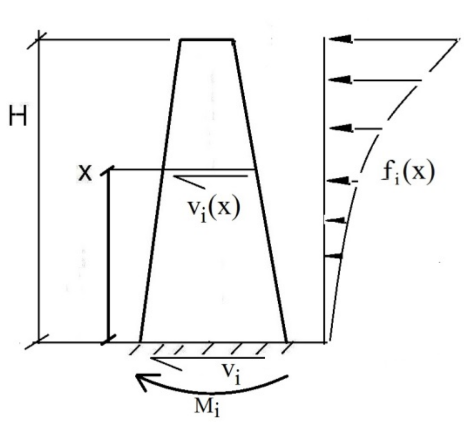

Figure 3. Equivalent lateral load $f_i(x)$, shear force acting on the cross section of the structure $V_i(x)$, base shear force V_i and base overturning moment M_i

11. COMBINATION OF THE VIBRATION FREQUENCIES

11.1 Foppl and Southwell theories

According to replacement beam theory, the global and local bending stiffness as well as shear stiffness of original structure are calculated and then are substituted into relations developed in preceding sections. The result is the frequencies of global and local bending beams, shear beam and Timoshenko beam. These frequencies can be combined according to "Foppl" and "Southwell" theories [11,12]. "Foppl" theory is as follows:

$$\frac{1}{\left(\omega_m^T\right)^2} = \frac{1}{\left(\omega_m^{B_0}\right)^2} + \frac{1}{\left(\omega_m^S\right)^2} \tag{35}$$

Therefore, if the original structure has only global bending and shear stiffness, it vibrates as an equivalent Timoshenko beam and its vibration frequencies can be calculated using "Foppl" theory or Timoshenko beam equation (section 8). Furthermore, if the original structure has local bending stiffness also,its vibration frequencies (ω), can be calculated by combination of the vibration frequencies of Timoshenko beam (ω_m^T) and vibration frequencies of the beam with local bending stiffness ($\omega_m^{B_1}$) using"Southwell" theory:

$$\omega^2 = \left(\omega_m^T\right)^2 + \left(\omega_m^{B_1}\right)^2 \tag{36}$$

11.2 Dunkerley's theorem

When there is a lumped mass at free end of the beam, we can use obtained relations in sections 4 and 6 to calculate the vibration frequencies of the beam. But, first vibration frequency of such beam can also be calculated by Dunkerley's theorem. Dunkerley's theorem is expressed as [12]:

$$\begin{cases} \dfrac{1}{\left(\omega_{Mm}^B\right)^2} = \dfrac{1}{\left(\omega_M^B\right)^2} + \dfrac{1}{\left(\omega_m^B\right)^2} \\ \dfrac{1}{\left(\omega_{Mm}^S\right)^2} = \dfrac{1}{\left(\omega_M^S\right)^2} + \dfrac{1}{\left(\omega_m^S\right)^2} \end{cases} \tag{37}$$

ω_m^S , ω_M^S , ω_m^B and ω_M^B can be calculated by relations presented in sections 3, 4, 5 and 6, respectively. We can also use "Foppl" theory, to calculate vibration frequency of Timoshenko beam with lumped mass at free end ω_{Mm}^T by combination of $\omega_{Mm}^{B_0}$ and ω_{Mm}^S as follows:

$$\frac{1}{\left(\omega_{Mm}^T\right)^2} = \frac{1}{\left(\omega_{Mm}^{B_0}\right)^2} + \frac{1}{\left(\omega_{Mm}^S\right)^2} \tag{38}$$

Table. 1summarizes all various vibration frequencies which can be calculated using presented relations in this paper.

Table 1: all various vibration frequencies which can be calculated using presented relations in this paper

ω_m^S	Section 3	ω_m^T	Section 8
ω_{Mm}^S *and* ω_M^S	Section 4	ω_{Mm}^B	Section 11
ω_m^B	Section 5	ω_{Mm}^T	Section 11
ω_{Mm}^B *and* ω_M^B	Section 6	ω_{Mm}^S	Section 11

12.REPLACEMENT BEAM STIFFNESS

We consider n lateral load resisting subsystems for original structure. The k_{th} element has stiffnesses D_{0k}, S_k and D_{1k}. The stiffnesses of the replacement beam which replaces the n lateral load-resisting subsystems are denoted by D_0, S and D_1. These parameters are calculated as follows:

$$S=\pi^2\frac{B^3}{C^2},\qquad D_0=\frac{1}{\frac{C}{B^2}-\frac{C^2}{l_0^2B^3}},\qquad D_1=A-\frac{B^2}{C} \tag{39}$$

In which:

$$A=\sum_{k=1}^{n}\left(\frac{D_{0k}}{1+\frac{\pi^2D_{0k}}{l_0^2S_k}}+D_{1k}\right),\quad B=\sum_{k=1}^{n}\frac{D_{0k}}{\left(1+\frac{\pi^2D_{0k}}{l_0^2S_k}\right)^2}\times\frac{\pi^2D_{0k}}{S_k},\quad C=\sum_{k=1}^{n}\frac{D_{0k}}{\left(1+\frac{\pi^2D_{0k}}{l_0^2S_k}\right)^3}\times\left(\frac{\pi^2D_{0k}}{S_k}\right)^2 \tag{40}$$

Please seePotzta and Kollar [13] for details of calculations of D_{0k}, S_k and D_{1k} $(k=1,2,...n)$

13. NUMERICAL EXAMPLES

Three numerical examples are presented in this section to show applicability, efficiency, and accuracy of the presented method. In the presented examples, the vibration frequencies of tall structures with various lateral load-resisting systems are calculated.

13.1 Tall structure with wall-frame lateral load-resisting system

In this example, the first three vibration frequencies of a 30-story building with wall-frame lateral load-resisting system are calculated (Fig. 4). The building is assumed to have variable properties along the height. The results are compared to those obtained using available finite elementsoftware. Elastic modulus (E), shear modulus(G), story height(h), building height(L) and mass per unit length ($m(\xi)$) are given as:

$$E=2.5\times10^7\ \tfrac{KN}{m^2},\quad G=0.4E,\quad h=3\,m,\quad L=30\times3=90\,m,\quad m(\xi)=300\ \tfrac{ton}{m} \tag{41}$$

The properties of lateral load-resisting system of building are proposed in Table 2. The global bending stiffness(D_0), local bending stiffness(D_1) and shear stiffness(S) of replacement beam for first three modes are proposed in Table 3.

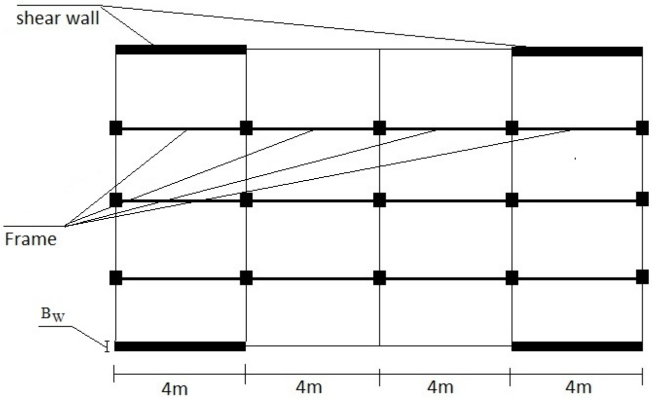

Figure 4. symmetrical plan of the example 13.1

Table 2: The lateral load-resisting system properties of building

Story Number	$1-5$	$5-10$	$10-15$	$15-25$	$25-30$	Unit
Inertial moment of frame I_f	5.208	3.417	2.133	1.251	0.675	$\times 10^{-3}\ m^4$
Cross sectional area of frame A_f	0.25	0.2025	0.16	0.1225	0.09	m^2
Shear walls width B_w	0.4	0.35	0.3	0.25	0.2	m

Table 3: Replacement stiffnesses of tall structure

mode	$i=1$		$i=2$		$i=3$		Unit
ξ	$\xi=0$	$\xi=1$	$\xi=0$	$\xi=1$	$\xi=0$	$\xi=1$	
D_0	30.044	10.785	25.741	4.2681	9.9274	1.1516	$\times 10^8\ KN-m^2$
D_1	21.001	10.498	18.73	9.3266	14.857	7.0552	$\times 10^7\ KN-m^2$
S	9.805	1.2746	10.646	1.6749	15.236	4.876	$\times 10^5\ KN$

Stiffness functions of the building are assumed as:

$$\begin{cases} D_0(\xi)=\alpha_0(\beta_0+\xi)^4 & (KN-m^2) \\ D_1(\xi)=\alpha_1(\beta_1+\xi)^4 & (KN-m^2) \\ S(\xi)=\alpha_2(\beta_2+\xi)^2 & (KN) \end{cases} \tag{42}$$

In which α_i and β_i ($i=0,1,2$) are unknown constants parameters which are determined through values presented in Table 3. The rotational inertia of replacement beam is calculated as follows:

$$r(\xi) = \sqrt{\frac{I(\xi)}{A(\xi)}} = \sqrt{\frac{kG}{E} \times \frac{D(\xi)}{S(\xi)}} \tag{43}$$

In which k is the correction factor of shear force. In this example $k = 1$ is adopted. This building has shear-bending stiffness, therefore it vibrates as a non-prismatic Timoshenko beam with shear and global bending stiffness which is supported laterally by a beam with local bending stiffness. The vibration frequencies of equivalent Timoshenko beam are calculated by two methods: 1. Solving the governing differential equation for vibration of Timoshenko beam, 2. Solving the governing differential equations for vibration of bending and shear beams and combination of their vibration frequencies using “Foppl” theory. By substituting functions $D_0(\xi)$ and $S(\xi)$ into relations obtained for Timoshenko beam, the first three vibration frequencies of Timoshenko beam ω_m^T are calculated. The results are presented in Table 4. Similarly, the first three vibration frequencies of bending beam and shear beam $\omega_m^{B_0}$, ω_m^S are calculated by substituting functions $D_0(\xi)$ and $S(\xi)$ intorelations obtained for them. According to “Foppl” theory, the first three vibration frequencies of equivalent Timoshenko beam are calculated by combination of vibration frequencies of bending and shear beams. These results are also proposed in Table 4.

Table 4: the first three vibration frequencies of equivalent Timoshenko beam by two methods

	mode	$i=1$	$i=2$	$i=3$
$\omega_m^T\ (\frac{rad}{\sec})$	solving Timoshenko beam equation	0.585	2.3587	5.282
	combination of bending and shear vibration frequencies	0.6493	1.9565	4.1759

By substitution of $D_1(\xi)$ into relations obtained for bending beam, the first three vibration frequencies corresponding to local bending stiffness are determined ($\omega_m^{B_1}$). The results are proposed in Table 5. Finally, we use “Southwell” theory to combine ω_m^T and $\omega_m^{B_1}$. The result is the vibration frequencies of original structure ω .The results are proposed in Table 6 and compared with those obtained using $SAP-2000$ software.

Table 5: The first three vibration frequencies of bending beam with local bending stiffness.

mode	$i=1$	$i=2$	$i=3$
$\omega_m^{B_1}\ (\frac{rad}{\sec})$	0.3399	1.8709	4.524

Table 6: The first three vibration frequencies of original structure

	mode	$i=1$	$i=2$	$i=3$
$\omega\ (\frac{rad}{\sec})$	Combination of ω_m^T (Timoshenko bam equation) and $\omega_m^{B_1}$	0.6766	3.011	6.9546
	Combination of $\omega_m^{B_0}$, ω_m^S and $\omega_m^{B_1}$	0.7329	2.7071	6.1567
	SAP-2000	0.6979	2.6424	6.1773

The presented approach can be used to determine the vibration frequencies of tall buildings with constant properties along the height. To achieve this, the properties of the replacement beam such as stiffness and mass per unit length are assumed to be constant along the height. We have calculated the first vibration frequency of bending and shear beams of the replacement beam presented in this example while it is assumed that the beam has constant stiffness that is identical to stiffness at $\xi = 0$. The results are proposed in Table 7.

Table 7: the first vibration frequency of bending and shear beams having constant properties along the height

	$\omega_m^{B_0}$	$\omega_m^{B_1}$	ω_m^{S}
Presented approach	1.3736	0.3631	0.997793
Tarjan and Kollar[12]	1.3752	0.3635	0.997793
ETABS	1.3736	0.3635	0.9951

13.2 Tall structure with frame and coupled shear walls systems

The numerical example proposed by Kaviani et. al [11] is investigated and the vibration frequency of building is calculated using presented approach in this paper. A 30-story building with a symmetrical plan is considered (Fig. 5). The story height is $h = 3.05\,m$. The lateral load-resisting systems of the building aretwo shear walls in the y-direction and a combination of a frame and two coupled shear walls in the z-direction. The vibration frequency of building in the z-direction is calculated in this example.

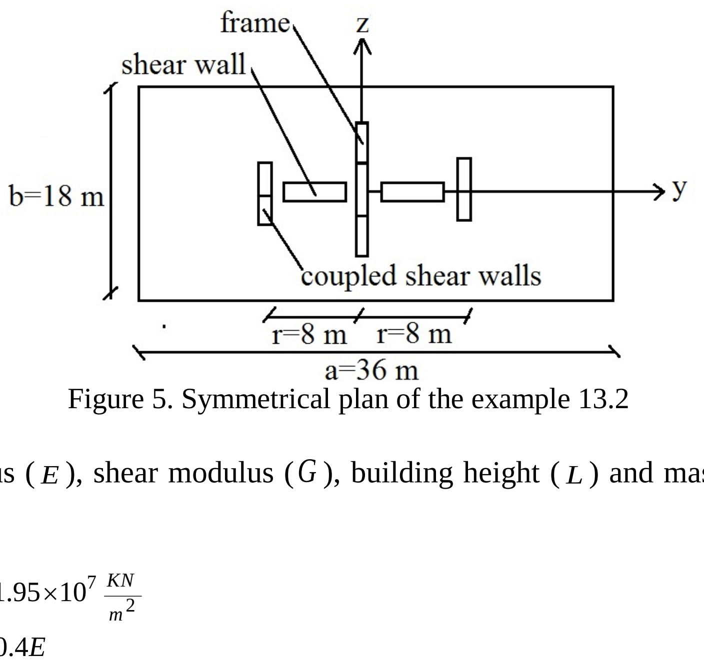

Figure 5. Symmetrical plan of the example 13.2

Elastic modulus (E), shear modulus (G), building height (L) and mass per unit length are given as:

$$\begin{cases} E = 1.95\times10^7\ \frac{KN}{m^2} \\ G = 0.4E \\ m = 3.058\times10^2\ \frac{ton}{m} \\ L = 30\times3.05 = 91.5\,m \end{cases} \tag{44}$$

The stiffness functions of replacement beam for first mode are calculated as:

$$\begin{cases} D_0(\xi) = 2220.27(\xi - 14.347)^4 & (KN - m^2) \\ D_1(\xi) = 62.079(\xi - 33.64)^4 & (KN - m^2) \\ S(\xi) = 699.71(\xi - 7.41)^2 & (KN) \end{cases} \tag{45}$$

The first vibration frequency of equivalent Timoshenko beam is calculated and proposed in Table 8. The first vibration frequency of original structure is calculated and compared to the one given by *ETABS* and Kaviani et al.[11]. The results are proposed in Table 9.

Table 8: The first vibration frequency of equivalent Timoshenko beam

	solving Timoshenko beam equation	combination of bending and shear vibration frequencies
$\omega_m^T \ (\frac{rad}{\sec})$	0.1674	0.1430

Table 9: The first vibration frequency of original structure

mode	Combination of ω_m^T (Timoshenko beam equation) and $\omega_m^{B_1}$	Combination of ω_m^S , $\omega_m^{B_0}$ and $\omega_m^{B_1}$	Kaviani et al. [11]	*ETABS*
$\omega \ (\frac{rad}{\sec})$	0.2698	0.2554	0.2630	0.2719

13.3 Tall structure with lumped mass at free end

The vibration frequencies of a tall structure with lumped mass at free end are calculated in this example. Height of the building and lumped mass at its free end are 50 m and 10000 Ton, respectively. The distribution of the global bending stiffness D_0 ,shear stiffness S and mass per unit length m along the height of the structure are as follows:

$$\begin{cases} D_0(\xi) = 175\times10^9(2-\xi)^4 \quad KN - m^2 \ , \quad 0\langle\xi\langle1 \\ S(\xi) = 6730\times10^6(2-\xi)^2 \quad KN \ , \quad 0\langle\xi\langle1 \\ m(\xi) = 2039.42(2-\xi)^2 \quad \frac{Ton}{m} \ , \quad 0\langle\xi\langle1 \end{cases} \tag{46}$$

The structure has global bending and shear stiffness only, therefore it can be modeled by an equivalent Timoshenko beam with a lumped mass at free end. The first five vibration frequencies for bending and shear beams with lumped mass at free end $\omega_{Mm}^{B_0}$,ω_{Mm}^{S} are calculated and proposed in Table 10. By combination of these frequencies according to “Foppl” theory, the vibration frequencies of equivalent Timoshenko beam ω_{Mm}^{T} are calculated. The results are presented in Table 11 and compared to results obtained from SAP-2000.

Table 10: The first five vibration frequencies of bending and shear beams with lumped mass at free end

	Mode number	1	2	3	4	5
$\omega_{Mm}^{B_0}$	Presented approach	29.746	125.495	318.873	615.27	1016.94
	SAP-2000 results	29.745	125.48	318.82	615.52	1016.7
	Mode number	1	2	3	4	5
ω_{Mm}^{S}	Presented approach	68.99	163.397	266.211	372.713	481.481
	SAP-2000 results	68.99	163.37	266.14	372.53	481.09

Table 11: The first five vibration frequencies of equivalent Timoshenko beam with lumped mass at free end

	Mode number	1	2	3	4	5
ω_{Mm}^{T}	Presented approach	27.315	99.5	204	318	435
	SAP-2000 results	27.47	94.35	193	305	422

According to Dunkerley's Theorem, we combine $\omega_{1m}^{B_0}$ and $\omega_{1M}^{B_0}$ to calculate $\omega_{1Mm}^{B_0}$. Also, using this theory, we combine ω_{1m}^{S} and ω_{1M}^{S} to calculate ω_{1Mm}^{S}. Lastly, using "Foppl" theory, we combine $\omega_{1Mm}^{B_0}$ and ω_{1Mm}^{S} to calculate ω_{1Mm}^{T}. The results are presented in Tables 12 and 13.

Table 12: The first vibration frequency of bending and shear beams

$\omega_{1m}^{B_0}$	Presented approach	34.275	$\omega_{1M}^{B_0}$	Presented approach	57.962
	SAP-2000 result	34.272		SAP-2000 result	57.966
ω_{1m}^{S}	Presented approach	73.712	ω_{1M}^{S}	Presented approach	164.083
	SAP-2000 result	73.708		SAP-2000 result	164.08

Table 13: The first vibration frequency of bendingand shear beams with lumped mass calculated using Dunkerley's Theorem and the first vibration frequency of equivalent Timoshenko beam with lumped mass

$\omega_{1Mm}^{B_0}$	Presented approach	29.502	ω_{1Mm}^{T}	Presented approach	27.015
	SAP-2000 result	29.745			
ω_{1Mm}^{S}	Presented approach	67.238		SAP-2000 result	27.47
	SAP-2000 result	68.99			

For first five modes of the bending vibration with lumped mass at free end, the mode shapes, the distribution of horizontal loadsand shear forces are illustrated in Figs. 6-8, respectively. The maximum of the shear force arises at the bottom of the cantilever, but in the second mode there is another local maximum. This local maximum is calculated by evaluating the shear force at $x = 0.83H$. Tarjan andKollar [12] found location of this local maximum to be between $0.6H$ and $0.9H$.

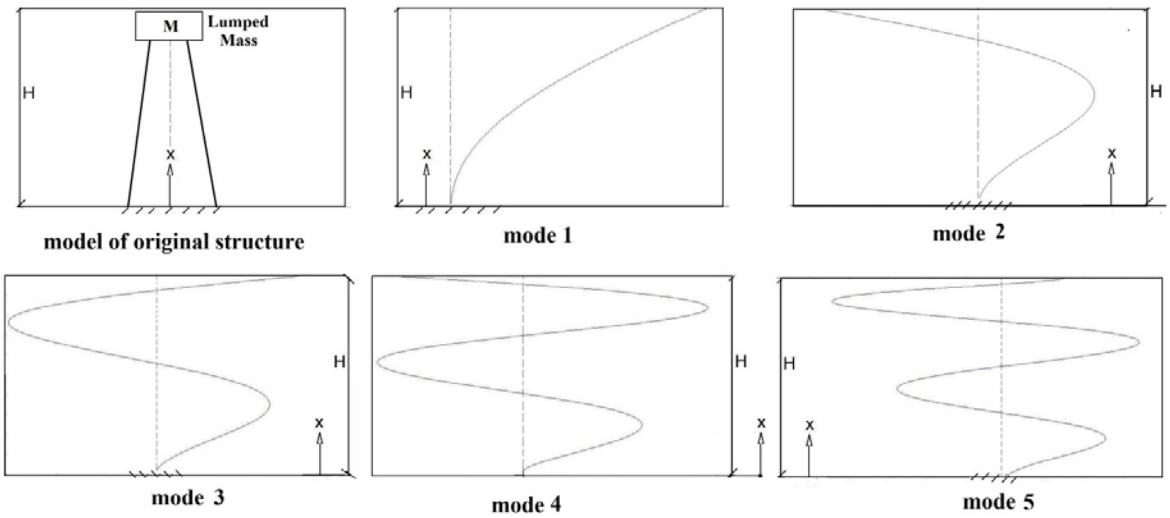

Figure 6. The first five mode shapes of the bending vibration with lumped mass

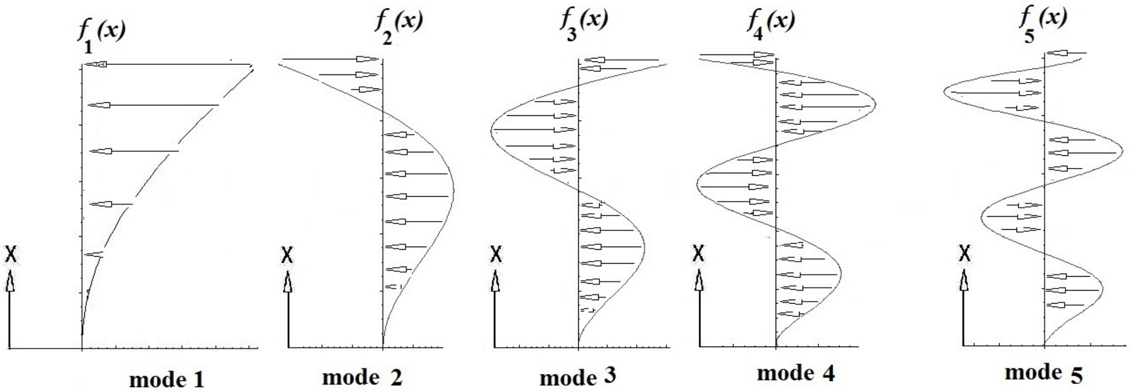

Figure 7. Equivalent lateral loads for first five modes of the bending vibration

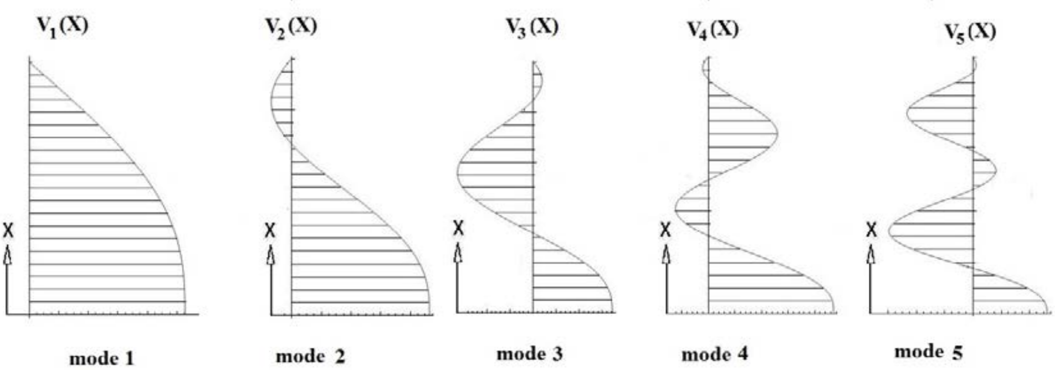

Figure 8. Shear forces acting on the cross section of the beam for first five modes

14. CONCLUSION

The vibration frequencies of tall structures with variable properties and lumped mass at free end are analytically and numerically investigated. According to replacement beam theory, the tall structure with various lateral load-resisting systems is modeled by an equivalent cantilever Timoshenko beam with global bending and shear stiffness which is supported laterally by a beam with local bending stiffness. The governing differential equations for free vibration of non-prismatic bending and shear beams as well as Timoshenko beam are solved and corresponding vibration frequencies are calculated. The vibration frequencies of original structure are calculated by combination of the obtained vibration frequencies. After calculation of the vibration frequencies, the presented analytical method is used to calculate the mode shapes, equivalent horizontal forces, shear forces and base overturning moment of the structure. "Foppl", "Southwell" and "Dunkerley" theories are used to combine the vibration frequencies. The accuracy of the presented approach is verified by three numerical examples and compared to other references.

REFERENCES

1. Kazaz İ, Gülkan P. An alternative frame-shear wall model: continuum formulation, *The Structural Design of Tall and Special Buildings*,**21**(2012) 524-42.
2. Park YK, Kim HS, Lee DG. Efficient structural analysis of wall–frame structures, *The Structural Design of Tall and Special Buildings*, **23**(2014) 740-59.
3. Rahgozar R, Mahmoudzadeh Z, Malekinejad M. Dynamic analysis of combined system of framed tube and shear walls by Galerkin method using B-spline functions, *The Structural Design of Tall and Special Buildings*, DOI: 10.1002/tal.1201, 2014.
4. Kamgar R, Saadatpour MM. A simple mathematical model for free vibration analysis of combined system consisting of framed tube, shear core, belt truss and outrigger system with geometrical discontinuities, *Journal of Applied Mathematical Modeling*, **36**(2012) 4918-30.
5. Malekinejad M, Rahgozar R. A simple analytic method for computing the natural frequencies and mode shapes of tall buildings, *Journal of Applied Mathematical Modeling*, **36**(2012) 3419-32.
6. Malekinejad M, Rahgozar R. An analytical model for dynamic response analysis of tubular tall buildings, *The Structural Design of Tall and Special Buildings*, **23**(2014) 67-80.
7. Zalka KA. Torsional analysis of multi-storey building structures under horizontal load, *The Structural Design of Tall and Special Buildings*, **22**(2013) 126-43.
8. Mohammadnejad M, Saffari H, Bagheripour MH. An analytical approach to vibration analysis of beams with variable properties, *Arabian Journal for Science and Engineering*, **39**(2014) 2561-72.

9. Saffari H, Mohammadnejad M, Bagheripour MH. Free vibration analysis of non-prismatic beams under variable axial forces, *Structural Engineering and Mechanics, An International Journal*, **43**(2012) 561-82.
10. Lee WH. Free vibration analysis for tube-in-tube tall buildings, *Journal of Sound and Vibration*, **303**(2007) 287-304.
11. Kaviani P, Rahgozar R, Saffari H. Approximate analysis of tall buildings using sandwich beam models with variable cross section, *The Structural Design of Tall and Special Buildings*, **17**(2008) 401-18.
12. Tarjan G, Kollar LP. Approximate analysis of building structures with identical stories subjected to earthquakes, *International Journal of Solids and Structures*, **41**(2004) 1411-33.
13. Potzta G, Kollar LP. Analysis of building structures by replacement sandwich beams, *International Journal of Solid and Structures*, **40**(2003) 535-53.
14. Li QS, Fang JQ, Jeary AP. Free vibration analysis of cantilevered tall structures under various axial loads, *Journal of Engineering Structures*, **22**(2000) 525-34.
15. Lee J, Bang M, Kim JY. An analytical model for high-rise wall-frame structures with outriggers, *The Structural Design of Tall and Special Buildings*, **17**(2008) 839-51.
16. Kwan AKH. Simple method for approximate analysis of framed tube structures, *Journal of Structural Engineering, ASCE*, **120**(1994) 1221-39.
17. Stafford Smith B, Coull A. *Tall Building Structures: Analysis and Design*, Wiley, New York, 1991.
18. Taranath BS. *Structural Analysis and Design of Tall Buildings*, McGraw-Hill, New York, 1988.
19. Bozdogan KB. An approximate method for static and dynamic analysis of symmetric wall-frame buildings, *The Structural Design of Tall and Special Buildings*, **18**(2009)279-90.
20. Li QS. A new exact approach for determining natural frequencies and mode shapes of non-uniform shear beams with arbitrary distribution of mass or stiffness, *International Journal of Solids and Structures*, **37**(2000)5123-41.
21. Clough RW, Penzien J. *Dynamics of Structures*, McGraw-Hill Book Company, New York, 1975.
22. Chopra AK. *Dynamics of Structures Theory and Application to Earthquake Engineering*, Prentice Hall, New Jersey, 1995.