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EXPANSION PROCESS IN MATHEMATICS AND STRUCTURAL MECHANICS

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ABSTRACT

The idea of expansion has a long history in mathematics and structural mechanics. In mathematics, Whitehead [1] used an expansion process for the characterization of topological spaces. In structural mechanics, Müller Breslau and Henneberg [2,3] employed simple expansion for the classification of trusses. In this paper, the latter idea is generalized and applied to the calculation of degree of statical indeterminacy (DSI) and degree of kinematical indeterminacy (DKI) of different types of skeletal structures, such as rigid-jointed planar and space frames, pin-jointed planar trusses and ball-jointed space trusses. Such a calculation not only simplifies the evaluation of DSI and DKI, but it also provides an insight to the problem of the formation of sparse statical and kinematical basis matrices required for efficient analysis of structures by the force method and displacement approach, respectively.

Keywords: Expansion; union intersection theorem; graph theory; structures; degree of statical indeterminacy; degree of kinematical indeterminacy; force method; displacement method.

1. INTRODUCTION

In the analysis of skeletal structures, three different properties are encountered, which can be classified as topological, geometrical and material. Separate study of these properties results in a considerable simplification in the analysis and leads to a clear understanding of the structural behaviour. This paper is confined to partial study of the topological properties of skeletal structures, since both displacement and force methods require such a study at the beginning of the analysis. The number of equations to be solved in the two methods may differ widely for the same structure. This number depends on the size of flexibility and stiffness matrices, which are the same as the DSI and the DKI of a structure, respectively. Obviously, the method which leads to the required results with the least amount of effort,

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should be used for the analysis of a given structure. Thus, the comparison of the numbers of DSI and DKI may be the main deciding criterion for selecting the method of analysis.

For determining the DSI and DKI of structures, numerous formulae, depending on the kinds of member or types of joint, have been given, e.g. Refs. [4,5]. The use of these classical formulae, in general, requires counting the number of members and joints, which becomes a tedious process for multi-member and/or complex pattern structures. This counting process provides no additional information about their connectivity properties.

Henderson and Bickley [6] related the DSI of a rigid-jointed frame to the first Betti number (cyclomatic number) of its graph model S. This was an important achievement, since a topological invariant of a graph was related to an essential mechanical property of the corresponding structure. Generalizing the Betti´s number to a linear function and using an expansion process, Kaveh developed a general method for determining the DSI and DKI of all types of skeletal structures [7]. Special methods have also been developed to transform the topological properties of space structures to those of their planar drawings to simplify the calculation of their DSI, Ref. [8-10].

In this paper, the latter idea is generalized and applied to the calculation of DSI and DKI of different types of structures, such as rigid-jointed planar and space frames, pin-jointed planar trusses and ball-jointed space trusses. Such a calculation not only simplifies the evaluation of DSI and DKI, but it also provides an insight to the problem of forming static and kinematic basis leading to highly sparse structural matrices.

It should be noted that various methods for determining the degree of static indeterminacy of structures are a by-product of the general methods developed here. The method of expansion and its control at each step, using the intersection theorem of this paper, provides a powerful tool for further studies in the field of structural analysis.

2. BASIC DEFINITIONS AND GRAPH MODELS OF STRUCTURES

2.1 Basic definitions from graph theory

A *graph* S consists of a set N(S) of elements called *nodes* (vertices or points) and a set M(S) of elements called *members* (edges or arcs) together with a relation of incidence which associated with each member a pair of nodes, called its *ends*. The connectivity properties of a skeletal structure can simply be transformed into that of a graph S; the joints and the members of the structure correspond to the nodes and the edges of S, respectively.

Two or more members joining the same pair of nodes are known as *multiple members*, and a member joining a node to itself is called a *loop*. A graph with no loops and multiple members is called a *simple graph*. If N(S) and M(S) are countable sets, then the corresponding graph S is finite. A graph S_i is a *subgraph* of S if $N(S_i) \subseteq N(S)$, $M(S_i) \subseteq M(S)$, and each member of S_i has the same end nodes as in S. A *path* of S is a finite sequence $P_i = \{n_0, m_1, n_1, ..., m_p, n_p\}$ whose terms are alternately distinct nodes n_i and distinct members m_i of S for $1 \leq i \leq p$, and n_{i-1} and n_i are the two ends of m_i.

Two nodes n_i and n_i are said to be *connected* in S if there exists a path between these nodes. A graph S is called *connected* if all pairs of its nodes are connected. A *component* of S is a maximal connected subgraph, i.e. it is not a subgraph of any other connected subgraph of S. A graph is *2-connected* if it remains connected when one of its member is removed. A *cycle* is a path (n_0, m_1, n_1,..., m_p, n_p) for which $n_0 = n_p$ and $p \geq 3$. A *tree* T of S is a connected subgraph which contains no cycle. If a tree contains all the nodes of S it is called a *spanning tree* of S.

2.2 Graph model of a structure

The mathematical model S of a structure is considered to be a finite, connected graph. There is a one-to-one correspondence between the elements of the structure and the members of S. There is also a one-to-one correspondence between the joints and the nodes of S, except for the support joints. For frame structures, two different groups of modelling are considered. The first group is suitable for calculating the DSI and DKI of structures, and the second group is more appropriate for analysis. For a frame structure, all the support joints are identified as a datum (ground) node in the first group of model, and in the second group, all such joints are connected by an artificial tree. Truss structures are assumed to be supported in a statically determinate fashion, and the effect of additional supports can easily be included in calculating the DSI and DKI of the corresponding structures.

3. EXPANSION PROCESS FOR DETERMINING DSI OF A STRUCTURE

The *degree of kinematic indeterminacy* of a structure is the number of independent displacement components (translations and rotations) required for describing a general state of deformation of the structure. The degree of kinematic indeterminacy is also referred to as the *total degrees of freedom* of the structure. On the other hand, the *degree of static indeterminacy* (redundancy) of a structure is the number of independent force components (forces and moments) required for describing a general equilibrium state of the structure. The DSI of a structure can be obtained by subtracting the number of independent equilibrium equations from the number of its unknown forces.

3.1 Classical formulae

Formulae for calculating the DSI and DKI of various skeletal structures can be found in textbooks on structural mechanics, e.g. the DSI and DKI of a planar truss, denoted by $\gamma(S)$ and $\eta(S)$, respectively, can be calculated from,

$$\gamma(S) = M(S) - 2N(S) + 3, \quad (1)$$

$$\eta(S) = 2N(S) - 3, \quad (2)$$

where S is supported in a statically determinate fashion (internal indeterminacy). For extra supports (external indeterminacy), $\gamma(S)$ and $\eta(S)$ should be adjusted

Similar formulae are available for space trusses as:

$$\gamma(S) = M(S) - 3N(S) + 6. \quad (3)$$

$$\eta(S) = 3N(S) - 6, \quad (4)$$

For planar and space frames, the classical formulae are given as:

$$\gamma(S) = \alpha\,[M(S) - N(S) + 1], \tag{5}$$
$$\eta(S) = \alpha[N(S) - 1], \tag{6}$$

where all supports are modelled as a datum (ground) node, and $\alpha = 3$ or 6 for planar and space frames, respectively.

All these formulae require counting a great number of members and nodes, which makes their application impractical for multi-member and complex pattern structures. These numbers provide only a limited amount of information about the connectivity properties of structures. In order to obtain additional information, the methods developed in the following sections will be utilised:

3.2 A unifying function

All the existing formulae for determining the DKI and DSI have a common feature, which is their linearity with respect to M(S) and N(S). Therefore, a general unifying function can be considered as:

$$\nu(S) = aM(S) + bN(S) + c\nu_0(S), \tag{7}$$

where M(S), N(S) and $\nu_0(S)$ are the numbers of members, nodes and components of S, respectively. Here, $\nu_0(S)$ is the same zero Betti number $b_0(S)$ of the graph model. The coefficients a, b and c are integer numbers depending on both the type of the corresponding structure and the property that the function is expected to represent. For example, $\nu(S)$ with appropriate values for a, b and c may describe the DKI or DSI of certain types of skeletal structures, Table 1. For a = 1, b = - 1 and c = 1, $\nu(S)$ becomes the first Betti number $b_1(S)$ of S.

Table 1: The coefficients of classical formulae for different types of skeletal structures.

Type of structure	ν(S)	a	b	c
Plane frames	DKI	0	+3	–3
	DSI	+3	–3	+3
Space frames	DKI	0	+6	–6
	DSI	+6	–6	+6
Plane trusses	DKI	0	+2	–3
	DSI	+1	–2	+3
Space trusses	DKI	0	+3	–6
	DSI	+1	–3	+6

The above table can be extended by considering other skeletal structures and higher dimensional finite elements. The functions can also be representative of other properties of different non-structural models.

3.3 An expansion process

An expansion process, in its simplest form, has been used by Müller-Breslau [2] for re-forming structural models, such as simple planar and space trusses. In his expansion process, the properties of typical subgraphs, selected in each step to be joined to the previously expanded subgraph, guarantee the determinacy of the simple truss. These subgraphs consist of two and three concurrent bars for planar and space trusses, respectively. An interesting aspect of the expanded truss model is that if the joints add in k steps are considered in a reverse order of expansion, the bar forces can be found with 2 equations with 2 unknowns at a time for planar trusses and 3 equations with 3 unknowns for space trusses. This ordering leads to equilibrium equations of special pattern. Though a general truss has normally additional members, however, identifying a submodel with this property may provide useful information about the remaining part of the model for a more efficient solution of the corresponding equations. Such information may also help in studying the geometric stability or performing topology optimization of truss structures.

The idea can be extended to other types of structure, and more general subgraphs can be considered for addition at each step of the expansion process.

$$S_1 = S^1 \to S^2 \to S^3 \to \ldots \to S^{q)} = S, \tag{8}$$

where $S^k = \bigcup_{i=1}^{k} S_i$. We define the intersection of S^k and S_{k+1} as $A_{k+1} = S^k \cap S_{k+1}$.

In the above expansion process S_i can be selected as a repeated module of S, a cycle, a planar subgraph, and a subgraph with prescribed connectivity properties.

3.4 An intersection theorem

In a general expansion process, a subgraph S_i may be joined to another subgraph S_j in an arbitrary manner. For example, $\nu(S_i)$ or $\nu(S_j)$ may have any arbitrary value and the union $S_i \cup S_j$ may be a connected or a disjoint subgraph. The intersection $S_i \cap S_j$ may also be connected or disjoint. It is important to find the properties of $S_1 \cup S_2$ having the properties of S_1, S_2 and $S_1 \cap S_2$. This enables one to control the process of the considered expansion.

The following theorem provides a correct calculation of the properties of $S_i \cup S_j$. In order to have the formula in its general form, q subgraphs are considered in place of two subgraphs.

Theorem (Kaveh [7]): Let S be the union of q subgraphs S_1, S_2, S_3, ... ,S_q with the following functions being defined:

$$\nu(S) = aM(S) + bN(S) + c\,\nu_0(S),$$

$$\nu(S_i) = aM(S_i) + bN(S_i) + c\nu_0(S_i) \qquad i = 1,2,...,q,$$
$$\nu(A_i) = aM(A_i) + bN(A_i) + c\nu_0(A_i) \qquad i = 2,3,...,q,$$

where $A_i = S^{i-1} \cap S_i$ and $S^i = S_1 \cup S_2 \cup \ldots \cup S_i$. Then:

$$[\nu(S) - c\,\nu_0(S)] = \sum_{i=1}^{q}[\nu(S_i) - c\nu_0(S_i)] - \sum_{i=2}^{q}[\nu(A_i) - c\nu_0(A_i)] \tag{9}$$

For proof, the interested reader may refer to Kaveh [9,10].

Special Case: If S and each of its subgraphs considered for expansion (S_i for i = 1,...,q) are non-disjoint (connected), then Eq. (9) can be simplified as:

$$\nu(S) = \sum_{i=1}^{q} \nu(S_i) - \sum_{i=2}^{q} \overline{\nu}(A_i)\,, \tag{10}$$

where $\overline{\nu}(A_i) = aM(A_i) + bN(A_i) + c$.

For calculating the DSI of a multi-member structure, one normally selects a repeated unit of the structure and joins these units sequentially in a connected form. Therefore, Eq. (10) can be applied in place of Eq. (9) to obtain the overall property of the structure.

3.5 A method for determining the DSI or DKI of structures

Let S be the union of its repeated and/or simple pattern subgraphs S_i (i=1,...,q). Calculate the DSI of each subgraph, using the appropriate coefficients from Table 1. Now perform the union-intersection method with the following steps:

Step 1: Join S_1 to S_2 to form $S^2 = S_1 \cup S_2$, and calculate the DSI or DKI of their intersection $A_2 = S_1 \cap S_2$. The value of $\nu(S^2)$ can be found using Eq. (9) or Eq. (10), as appropriate.

Step 2: Join S_3 to S^2 to obtain $S^3 = S^2 \cup S_3$, and determine the DKI (degree of kinematical indeterminacy) or DSI of $A_3 = S^2 \cap S_3$. Similarly to Step 1, calculate $\gamma(S^3)$.

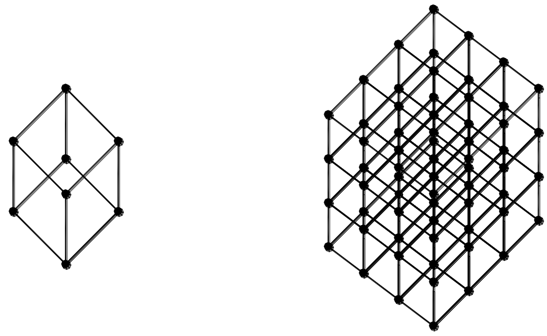
Step k: Subsequently join S_{k+1} to S^k, calculating the DSI of $A_{k+1} = S^k \cap S_{k+1}$ and evaluating the magnitude of $\nu(S^{k+1})$.

Repeat Step k until the entire structural model $S = \bigcup_{i=1}^{q} S_i$ has been reformed and its DSI determined.

In the above expansion process, the value of q depends on the properties of the substructures (subgraphs) which are considered for reforming S. These subgraphs have either simple patterns for which $\gamma(S_i)$ can easily be calculated, or the DSIs of which are already known.

In the process of expansion, if an intersection A_i itself has a complex pattern, further refinement is also possible; i.e. the intersection can be considered as the union of simpler subgraphs.

Example: Let S be the graph model of a space frame. This graph can be considered as 27 subgraphs S_i as shown in Fig. 1(a), connected to each other to form a graph $S = \bigcup_{i=1}^{27} S_i$. The interfaces of S_i (i=1,...,27) are shown in Fig. 1(b), in which some of the members are omitted for the sake of clarity.

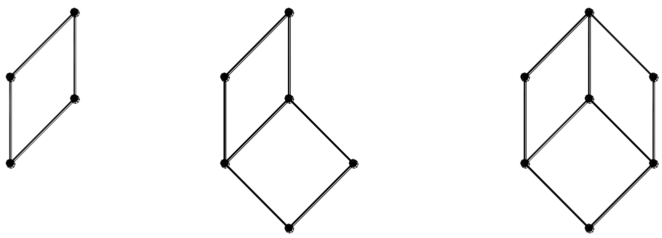


(a) A subgraph S_i of S. (b) $S = \bigcup_{i=1}^{27} S_i$ without some of its members.

Figure 1. A space structure S.

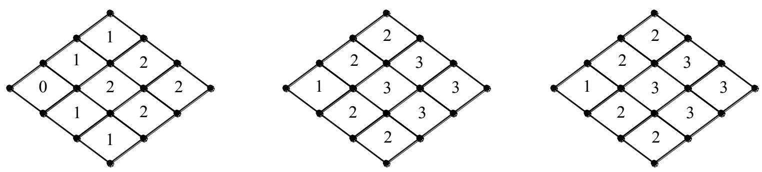
The expansion process consists of joining 27 subgraphs S_i one at a time. In this process, the selected subgraphs can have three different types of intersection, which are shown in Fig. 2(a). In order to simplify the counting and the recognition of the types of interfaces, S is re-formed storey by storey. For the first storey, a 3×3 table is used to show the types of intersections occurring in the process of expansion. The numbers on each box designate the type of intersection, Fig. 2(b). Similar boxes are used for the second storey and the third storey of S, Fig. 2(b).

Thus, there exist 6 intersections of type A_i^1, 12 intersections of type A_i^2 and 8 intersections of type A_i^3.



Type 1- A_i^1 Type 2- A_i^2 Type 3- A_i^3

(a) Three different types of intersections



First floor Second floor Third floor

(b) Types of intersection after the completion of each storey.

Figure 2. Intersections and their types

Since each S_i is a connected subgraph, and in the process of expansion, S_i is kept connected, a simplified Eq. (9) can be employed:

$$\gamma(S) = \sum_{i=1}^{27}\gamma(S_i) - \sum_{i=2}^{27}\bar{\gamma}(A_i).$$

As previously shown:

$$\sum_{i=2}^{27}\bar{\gamma}(A_i) = \sum_{i=2}^{7}\bar{\gamma}(A_i^1) + \sum_{i=8}^{19}\bar{\gamma}(A_i^2) + \sum_{i=20}^{27}\bar{\gamma}(A_i^3).$$

The intersections A_i^2 and A_i^3 can be decomposed as:

$$A_i^2 = A_i^1 \cup A_i^1 \qquad \text{and} \qquad A_i^3 = A_i^2 \cup A_i^1.$$

The DSI of S can now be calculated as follows:

$\gamma(S_i) = 6(12 - 8 + 1) = 6\times 5 = 30.$

Using Eq. (5),

$\bar{\gamma}(A_i^1) = 6(4 - 4 + 1) = 6,$

$\bar{\gamma}(A_i^2) = 6\times 1 + 6\times 1 - 6\times 0 = 12,$

$\bar{\gamma}(A_i^3) = 6\times 2 + 6\times 1 - 6\times 0 = 18.$

Hence

$\gamma(S) = 27(30) - [6(6)+12(12)+8(18)] = 486.$

The expansion process becomes very efficient for structures with repeated patterns. Counting is reduced considerably by this method. As an example, the use of the classical formula for finding the DSI of S in the above example requires counting 144 members and 64 nodes, which is a task involving possible errors.

4. OTHER APPLICATIONS OF DIFFERENT EXPANSION PROCESSES IN MECHANICS

Apart from finding the DSI of a structure, the expansion process provides a suitable tool for finding out the distribution of indeterminacy in the vicinity of the structural model. Such a distribution can be used in the formation of subgraphs on which S.E.Ss can be constructed. This also helps in designing more reliable structure. In fact a uniform distribution of indeterminacy make the structure more secure in the sense that if part of the structure is collapsed under an unexpected event, the remaining part of the model can redistribute the applied loads.

Expansion can also be utilized in topology optimization of truss structures where a preliminary structure with small number of members can simplify the optimization process. Such a structure can be a suitable substitute for the ground structure (originally defined as a clique graph) utilized in topology optimization which may contain many elements. This preliminary structure does not need to be statically determinate, and different criteria may be employed for its selection to achieve a suitable substitute for the ground structure. As an example, spanning all the nodes may be one criterion, or an even distribution for the degrees of the nodes can be another objective.

Expansion has also been used in matrix analysis of structures or finite element methods. In these methods an element by element expansion is used and the overall property of the structure is obtained by planting the stiffness matrix of each element in time (assembling process). Similarly in substructuring the expansion process is carried out substructure by substructure. Or in finite element method the expansion is performed hyper element by hyper element.

Ordering can also be considered as an expansion process during which nodal or element numbers are reorganised to provide a special patter for the model to simplify the solution of the corresponding equations (well-structured) [11-13].

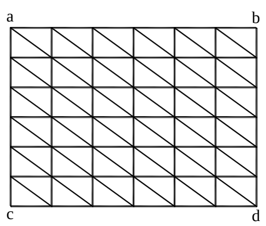
Graph product is also an expansion process where a subgraph expanded using some rules. These products have found many application in structural mechanics [14-17]. This expansion can be multi-stage expansion when higher dimension are also considered.

The new swift approaches expand first the regular part of the model and then add the remaining irregular parts. This method uses closed form solution for the regular part followed by a substructuring approach for completion of the remaining part of the model.

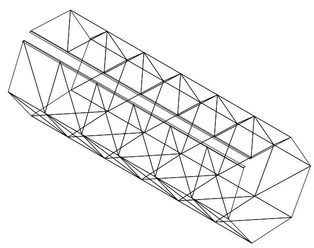
In some applications the properties of the intersection may be unimportant. An example of this case is in configuration processing, where the identical connectivity of the submodels is sufficient for generating the configuration [18].

5. IDENTIFICATION METHOD

This approach is also based on the union-intersection method and provides a simple means for finding the topological properties of a structural model S, after identifying two subgraphs of S. For example, consider a model as shown in Fig. 3(a). Identifying ab with cd, as in Fig. 3(b), results in a cylindrical space graph S^2. Identification of ac and bd, leads to a torus-like skeletal structure as depicted in Fig. 3(c).

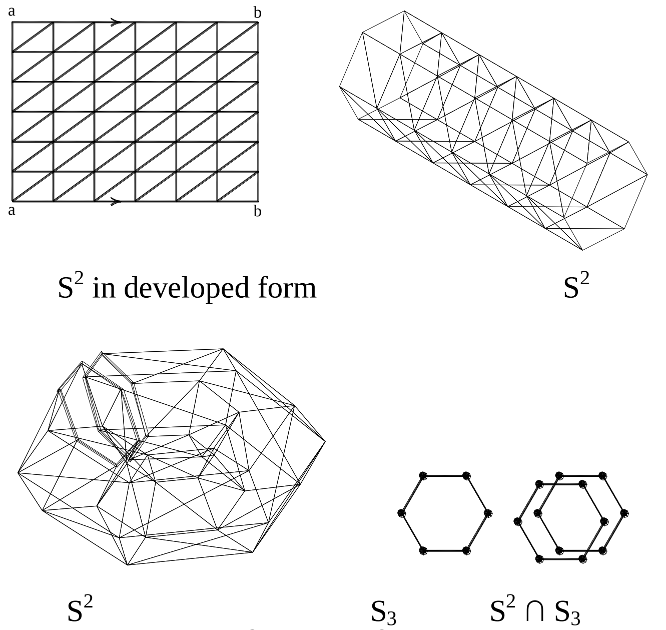


S_1 in developed form

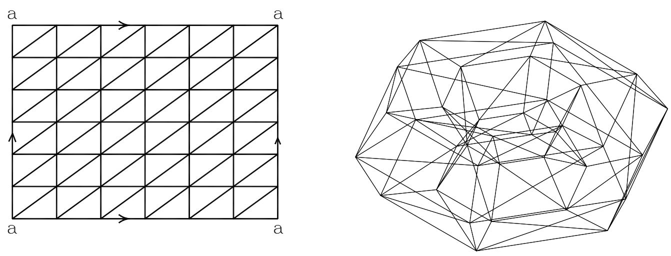


$S_1 = S^1$

(a) S_1, S_2 and $S^1 \cap S_2$.



(b) S^2, S_3 and $S^2 \cap S_3$.



(c) A torus-like structure $S^3 = S$.

Figure 3. Identifications in a structure.

The following equation can be employed in such an approach, which is similar to Eq. (9) of the previous section.

$$\nu(S^i \cup S_j) = \nu(S^i) + \nu(S_j) - \bar{\nu}(S^i \cap S_j). \tag{11}$$

In this relation, however, S_j is a subgraph of S^i through which the identification has been made. Obviously $S^i \cap S_j$ consists of two disjoint S_j.

Identification transforms a regular model into a circulant model. Eigensolution of circulant models is much easier than those of the regular ones. Such transformations simplify the modal and dynamic analysis of regular structures [19].

6. CONCLUDING REMARKS

In this paper, the process of expansion is demonstrated by means of calculating the DSI and DKI of skeletal structures. However, the main objective is by no means confined to finding these numbers, since there are enough formulae for this purpose. The manner in which these calculations are performed provides us with efficient approaches for optimal analysis of structures. The expansion process can easily be applied to configuration processing in which only repeated subgraphs are generated using functions such as translation, rotation, reflection and projection functions [14]. Considering subgraphs as cycles (γ-cycles) and imposing a suitable admissibility function on the intersection provides us with a powerful algorithm for the formation of suboptimal cycle bases (generalized cycle bases) for optimal force method of structural analysis [20-23]. This process can also be applied to finite element analysis by pure force method, where the formation of localized null bases, lead to highly sparse flexibility matrices [24].

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